THE ASTROPHYSICAL JOURNAL

AN INTERNATIONAL REVIEW OF SPECTROSCOPY AND ASTRONOMICAL PHYSICS

VOLUME 92 NOVEMBER 1940 NUMBER 3

THE INTERNAL TEMPERATURE OF WHITE DWARF STARS*

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ABSTRACT

The state of matter in the interior of a white dwarf star is investigated in detail. The main part of this paper is concerned with the calculation of the temperature distribution in the interior. For this purpose accurate expressions for the radiative and conductive opacity are derived, both for the nondegenerate envelope and the degenerate core of the stars. It is shown that the energy transport is chiefly by radiation in the envelope and by electronic conduction in the core. Furthermore, the free electron density is investigated, and it is found that in the transition region between degeneracy and nondegeneracy the atoms are only partially ionized, whereas both in the envelope and in the core the ionization is almost complete (in the envelope, temperature ionization; in the core, pressure ionization).

Quantitative calculations are performed for the two well-known white dwarfs, Sirius B and 40 Eridani B; of the two the observational material is better for Sirius B, so that most attention is paid to this star. The extension of the nondegenerate envelope and the internal temperature are calculated for several widely different assumptions regarding the chemical composition of the star as well as possible errors in the observed luminosity L and the radius R. The depth h of the envelope of Sirius B is about $6 \cdot 10^7$ cm, when the observed values of R, L and mass M are used and the star is assumed to consist exclusively of "Russell mixture." For a pure helium star h is even smaller, and, if the theoretical value for the radius is assumed, h is only $8 \cdot 10^6$ cm. Thus, the envelope constitutes only a very small fraction of the total radius of the star ($\approx 10^9$ cm).

The central temperature T_c of Sirius B is $\approx 15{,}000{,}000°$ for pure Russell mixture and is almost independent of the assumed radius; moreover, the degenerate core is practically isothermal. At this temperature and at the high densities of 10^5 to $3 \cdot 10^7$ gm/cm³ which prevail in the core, nuclear reactions between protons and other nuclei would go at a very rapid rate. Since the observed luminosity is very small, we must conclude that no appreciable number of protons is present inside of white dwarfs. If Sirius B contains carbon and nitrogen in the same abundance as do main-sequence stars, the hydrogen

* The major part of this work was done while the author was President White Fellow at Cornell University; preliminary results were reported with H. A. Bethe to the American Physical Society (cf. *Phys. Rev.*, **55**, 681, 1939, and **57**, 69, 1940).

concentration X_H cannot be more than $2 \cdot 10^{-8}$; and even if C and N are completely absent, X_H must be less than $2 \cdot 10^{-5}$, because the protons will still combine with each other. A lower reaction rate is obtained for a lower central temperature; for this purpose a pure helium star was investigated for which $T_c \approx 7 \cdot 10^{6°}$ was found; but even at this temperature the proton-proton reaction is so fast that X_H must be less than 10^{-3}.

For 40 Eridani B it turns out, with the assumption of Russell mixture and observed M, L, R, that the envelope has a depth of $3 \cdot 10^7$ cm, the central temperature a value of $30 \cdot 10^{6°}$, the hydrogen content an upper bound of $8 \cdot 10^{-5}$.

We must, therefore, conclude that no hydrogen is present in white dwarfs. Under these circumstances the radius of the star is uniquely determined by its mass in accordance with Chandrasekhar's theory of degenerate configurations. The theoretical radius of 40 Eridani B agrees well with the observed radius, but the radius of Sirius B is only $5.7 \cdot 10^8$ cm, as compared with an observed radius of $13.6 \cdot 10^8$ cm. This large discrepancy is very serious, since the observational value is considered good and there does not seem any way out from the standpoint of theory. Thus, no reasonable modifications of the theory (which, however, still keep the hydrogen content small) will appreciably increase the extent of the nondegenerate envelope and therefore the radius of the star. The only faint possibility would be to assume that Sirius B does not contain any light nuclei other than helium and hydrogen and that, moreover, the combination of two protons is much less probable than follows from well-established nuclear theory.

A further result of these investigations is the conclusion that the energy production of the white dwarfs is not due to thermonuclear reactions at all, but to gravitational contraction. With this source of energy it will take at least 10^8 years before the white dwarfs will become dark objects.

I. INTRODUCTION

Recent investigations[1,2,3,3a] into thermonuclear processes as the source of energy of main-sequence stars have proved quite conclusively that the carbon-cycle is responsible for the energy production in main-sequence stars brighter than the sun. For main-sequence stars fainter than the sun the evidence is less conclusive; calculations on the proton-proton reaction, using the Fermi β-decay theory, the Gamow-Teller selection rules, and the average of a set of empirically determined decay constants, seem to indicate that this reaction can account for the energy evolution of the fainter stars. For the sun itself theory predicts that the carbon-cycle and the proton-proton reaction give about equal contributions to the luminosity.[4]

On the other hand, it is well known that white dwarf stars differ from main-sequence stars of the same mass in two fundamental

[1] H. A. Bethe, *Phys. Rev.*, **55**, 434, 1939.

[2] H. A. Bethe and C. L. Critchfield, *Phys. Rev.*, **54**, 248, 1938.

[3] H. A. Bethe and R. E. Marshak, *Reports on Progress in Physics*, **6**, 1, London: Physical Society of London, 1939.

[3a] H. A. Bethe, *Ap. J.*, **92**, 118, 1940.

[4] R. E. Marshak and H. A. Bethe, *Phys. Rev.*, **56**, 210, 1939.

aspects: they have extremely low luminosities and exceedingly high densities (as a result of their very small radii), compared with the main-sequence stars. The high densities in the white dwarfs have been explained in terms of the theory of degenerate matter, and it has seemed likely that their low luminosities are due to an energy-production process other than that occurring in main-sequence stars.

In an attempt to settle the matter of the energy production and, moreover, to determine the modifications introduced into Chandrasekhar's theory of degenerate configurations[5] by the presence of finite temperatures, we have investigated in detail the temperature distribution in the interior of the white dwarfs. To make any headway at all it was necessary to obtain fairly accurate expressions for the radiative and conductive opacities not only under conditions of extreme degeneracy but also at the onset of degeneracy. Such expressions have been derived by the author,[6] and the results are summarized in section II. Similarly, the free-electron density, the pressure, and the Fermi energy are calculated for the transition region (between the radiative envelope and the degenerate core), and the results given in section III.

In section IV the star equations are integrated through the surface region of the two best-known white dwarfs, Sirius B and 40 Eridani B; both the temperature and the extent of the radiative envelope are found under different assumptions as to chemical composition, luminosity, and radius. In section V the integration of the equations of stellar equilibrium is carried through to the center, quite independent of any explicit assumptions concerning the energy-production process. The resulting temperature-density distribution is used to determine upper bounds on the hydrogen contents and radii of Sirius B and 40 Eridani B.

In the final section (sec. VI) the theory is compared with the observational material, and some conclusions are drawn regarding the sources of energy, the radii, and the evolutionary behavior of white dwarf stars.

[5] S. Chandrasekhar, *Introduction to the Theory of Stellar Structure*, esp. chaps. x and xi.

[6] R. E. Marshak, *Annals of New York Academy of Sciences*, 1940.

II. THE RADIATIVE AND CONDUCTIVE OPACITIES

Under conditions of nondegeneracy the radiative opacity K_R is given by Kramers' formula corrected by the insertion of a "guillotine" factor.[7] This formula may be written

$$K_R = \frac{6.0 \cdot 10^{17} X_R}{\tau T^2} e^{-\psi/kT} . \tag{1}$$

The quantity T is the temperature, k is the Boltzmann constant, X_R is the concentration of the Russell mixture of heavy elements, τ is the guillotine factor, and ψ is the negative of the Fermi energy ($\psi/kT \gg 0$ implies nondegeneracy, $\psi/kT \ll 0$ gives strong degeneracy, and $\psi/kT \approx 0$ holds for the intermediate region of incipient degeneracy). Equation (1) is valid for $\psi/kT \gg 0$; it is found[6] that the following expression is a good approximation for K_R in the regions $\psi/kT \approx 0$ and $\psi/kT \ll 0$:

$$K_R = \frac{6.0 \cdot 10^{17} X_R}{\tau T^2} \log_e \left[\frac{1 + e^{-\psi/kT}}{1 + e^{-\psi/kT - 7}} \right] . \tag{2}$$

In the limit $\psi/kT \to -\infty$, equation (2) goes over into the usual formula for the degenerate radiative opacity.[8] In the region $\psi/kT \approx 0$, $\tau \gg 1$, so that the contribution from the bound-free transitions is cut down considerably; for $\psi/kT \approx -5$, τ attains its maximum value of 196.5, and for larger negative values of ψ/kT only the free-free transitions contribute. Table 1 in reference 6 gives the calculated values of τ in the region $\psi/kT \approx 0$ for a pure Russell mixture.

In the interior of stars, particularly dense stars, energy is transported not solely by radiative processes but also by electronic conduction. If the energy current is Q, the thermal conductivity λ of the electrons is defined as

$$\lambda = \frac{-Q}{\frac{dT}{dx}} ,$$

[7] B. Strömgren, *Handb. d. Ap.*, **7**, 172, 1936.

[8] R. C. Majumdar, *A.N.*, **244**, 65, 1931.

with dT/dx the gradient of temperature T, along the direction x. The associated conductive opacity K_C is given by[9]

$$K_C = \frac{4acT^3}{3\rho\lambda}. \tag{3}$$

The quantity a is the Stefan-Boltzmann constant, c the velocity of light, and ρ the mass density.

The conductive opacity plays an important role in white dwarf stars. In the transition region between nondegeneracy and degeneracy $K_C \approx K_R$, while in the degenerate core the electrons transport most of the energy, so that $K_C \ll K_R$, and only the conductive part of the opacity is significant. This is due to the fact that for large ρ, $K_C \propto 1/\rho^2$, while K_R is independent of ρ.

The thermal conductivity λ (and hence K_C) has been calculated to a third approximation[10] on both the degenerate and the nondegenerate sides, in accordance with the usual procedure adopted in treating the problem in metals.[11] It is found[6] that for a concentration X_R of Russell mixture the degenerate conductive opacity is given by

$$K_C^D = \frac{7.82 \cdot 10^{-7} X_R I T^2}{(\rho\eta_e)^2[1 + 7.74 \cdot 10^{-10} W^2 + 6.22 \cdot 10^{-20} W^4]}; \tag{4}$$

we have written

$$\frac{\zeta_0}{kT} = \frac{2.98 \cdot 10^5}{W} \qquad \text{with} \qquad W = \frac{T}{(\rho\eta_e)^{2/3}}.$$

The quantity η_e is related to the free-electron density N_e, viz., $N_e = \rho\eta_e/m_H$; ζ_0 is the Fermi energy at absolute zero, and I is defined below.

An expression for the nondegenerate conductive opacity has also been derived.[6] We get

$$K_C^{ND} = \frac{2.85 \cdot 10^9 X_R I}{T} e^{-\psi/kT}. \tag{5}$$

[9] A. S. Eddington, *Internal Constitution of the Stars*.

[10] It was necessary to go to the third approximation, because over a large region ($\psi/kT = 0$ to $\psi/kT = -10$) the second approximation is quite bad.

[11] A. Sommerfeld and H. A. Bethe, *Handb. d. Phys.*, **24**, Part II, 532.

This formula is in its own range of application ($\psi/kT > 0$) more exact than equation (4) in its range ($\psi/kT < 0$).

Thus far we have not given any explicit form for I. It is found that

$$2I = \log_e \frac{2}{(1 - \cos \Theta_0)}, \tag{6}$$

where $\Theta_0 = \lambda/a$,[12] λ is the De Broglie wave length of the electron participating in the elastic collision, and a is the screening radius of the atom. Insertion of appropriate expressions for λ and a leads in the degenerate case to

$$I^D = \tfrac{1}{3}[5.9 + \tfrac{3}{2} \log_e (1 - 9.3 \cdot 10^{-12} W^2)]^{13} \tag{7}$$

and in the nondegenerate case to

$$I^{ND} = \frac{1}{3}\left[\log_e (3.92 \cdot 10^{-6} W^{3/2}) + \log_e \left(1 + \frac{2.15 \cdot 10^7}{W^{3/2}}\right)\right]. \tag{8}$$

These expressions are to be taken in conjunction with equations (4) and (5), respectively.

Figure 1 gives K_C as a function of ψ/kT for $T = 10 \cdot 10^{6\circ}$ C. The dotted curves show the values given by our formulae in regions where they are very bad approximations. Curves of this type permit one to estimate the inaccuracy introduced into K_C by the use of equations (4) and (5) for various values of ψ/kT and T.

III. THE FREE-ELECTRON DENSITY, PRESSURE, AND FERMI ENERGY

In order to find the gas pressure for given temperature and density it is necessary to know the free-electron density. As the mass density increases, the free-electron density at first decreases because of the greater occupation of the bound states of the atom. In the region $0 < \psi/kT < 4$ (which is on the nondegenerate side) we calculate the free-electron density by taking the number of free electrons per

[12] H. A. Bethe, *Handb. d. Phys.*, **24**, Part I, 497.

[13] Strictly speaking, this expression for I^D (and also K_C^D) is only valid for nonrelativistic degeneracy. However, when relativistic degeneracy is attained in a white dwarf, the absolute value of K_C^D is so small that relativistic corrections are not essential.

nucleus per quantum number n as $2n^2/(1 + e^{(\psi-\chi_n)/kT})$ with χ_n the energy of the nth bound shell.

The computation was done twice for a Russell mixture of elements; once values of χ_n were taken for the unscreened ionized atom and again for the screened nonionized atom. In the region of high

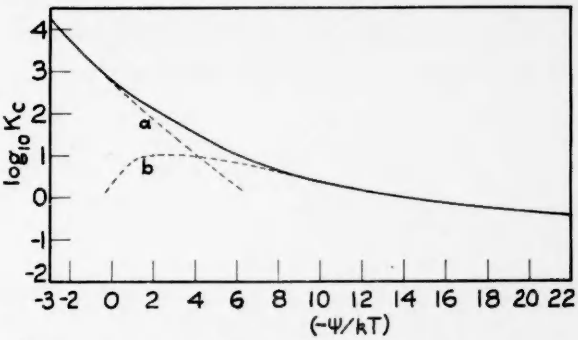

FIG. 1.—The heavy line gives conductive opacity, K_C, as a function of $(-\psi/kT)$ at a temperature of ten million degrees. The dotted lines (a) and (b) give the nondegenerate and degenerate approximations, respectively.

TABLE 1

THE FREE ELECTRON DENSITY (MULTIPLIED BY m_H/ρ)

T	ψ/kT				
	4	3	2	1	0
$1\cdot 10^6$	0.27	0.20	....	....	....
$2.5\cdot 10^6$	.33	.28	....	....	....
$4\cdot 10^6$	.40	.34	0.28	0.21	0.15
$6\cdot 10^6$	.43	.38	.32	.25	.18
$8\cdot 10^6$	.44	.41	.34	.27	.20
$10\cdot 10^6$	.45	.44	.35	.28	.21
$12\cdot 10^6$	0.46	0.45	0.35	0.28	0.21

temperatures on the nondegenerate side the assumption of an unscreened Coulomb field is the better approximation; in the region of low temperatures approaching mild degeneracy the assumption of screening takes account of the occupation of the lower bound states. Table 1 contains a reasonably interpolated set of values for η_e (the free electron density $N_e = \rho\eta_e/m_H$).

When $\psi/kT < 0$, some of the bound shells are pushed into the continuum, and the effective free-electron density is much larger than would be given by the Fermi method. This effect is taken into account in the following way.

The maximum of the radial-charge density of an electron in an atom is approximately at a distance $r = n'^2 a_0/(Z - s)$, where n' is the effective quantum number,[14] s is the screening constant which depends on the electron shell, and a_0 is the radius of the first Bohr

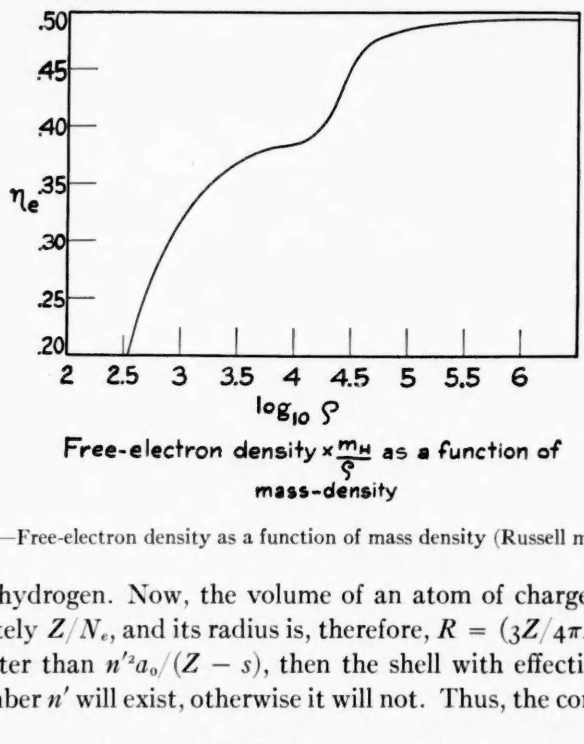

FIG. 2.—Free-electron density as a function of mass density (Russell mixture)

orbit of hydrogen. Now, the volume of an atom of charge Z is approximately Z/N_e, and its radius is, therefore, $R = (3Z/4\pi N_e)^{1/3}$. If R is greater than $n'^2 a_0/(Z - s)$, then the shell with effective quantum number n' will exist, otherwise it will not. Thus, the condition is

$$n' < 1 \cdot 18\,(\rho\eta_e)^{1/6}(Z - s)^{2/3}Z^{1/6}\,, \tag{9}$$

where we have used $N_e = \rho\eta_e/m_H$. We compute $(Z - s)^{2/3}Z^{1/6}$ for each of the elements of the Russell mixture and for the K, L, M, etc., shells when they are started and when they are complete—using Slater's semi-empirical values of s. We next find from equation (9) the values of $(\rho\eta_e)$, for which each shell of each element is just started

[14] Cf. J. C. Slater, *Phys. Rev.*, **36**, 57, 1930, where rules are given for finding n' and s.

and for which it is just complete. For a particular choice of $\rho\eta_e$ we can specify the number of free electrons per atom, and thus we can calculate η_e. Once η_e is known, the corresponding value of ρ can be determined. A graph of ρ against η_e, obtained in this self-consistent way, is given in Figure 2.

The free electrons are not alone in contributing to the gas pressure; there is also a contribution from the heavy particles, which increases as the atomic weight decreases. If we write for the number of nuclei per unit volume

$$N_{\text{heavy}} = \frac{\rho}{m_H}\eta_{\text{ heavy}} ,$$

then the total pressure is, in general, given by[15,16]

$$p_G = \eta_{\text{heavy}}\Re\rho T + \eta_e\Re\rho T \cdot \frac{V\left(\frac{\psi}{kT}, \frac{3}{2}\right)}{V\left(\frac{\psi}{kT}, \frac{1}{2}\right)} , \tag{10}$$

where

$$V\left(\frac{\psi}{kT}, m\right) = \frac{1}{\Gamma(m+1)}\int_0^\infty \frac{q^m dq}{e^{\psi/kT+q}+1} .$$

For

$$\frac{\psi}{kT} > 0 \quad \text{(non-degeneracy)} ,$$

$$V\left(\frac{\psi}{kT}, m\right) = e^{-\psi/kT} - \frac{e^{-2\psi/kT}}{2^{m+1}} + \frac{e^{-3\psi/kT}}{3^{m+1}} + \ldots ; \tag{11}$$

for

$$\frac{\psi}{kT} < 0 \quad \text{(degeneracy)} ,$$

$$\left.\begin{aligned} &V\left(\frac{\psi}{kT}, m\right) \\ &= \frac{\left(\frac{-\psi}{kT}\right)^{m+1}}{\Gamma(m+2)}\Bigg[1 + 2\sum_{\nu=2,4,6,\ldots}^{\infty}(1-2^{1-\nu})\xi_\nu\left(\frac{\psi}{kT}\right)^{-\nu}(m+1) \\ &\qquad \cdot m \ldots . (m-\nu+2)\Bigg] ; \end{aligned}\right\} \tag{12}$$

[15] R. C. Tolman, *Statistical Mechanics*. 2d ed.

[16] A. Sommerfeld, *Zs. f. Phys.*, **47**, 9, 1928.

ξ_ν is the Riemann zeta function of order ν. If we write

$$p_G = \frac{\Re}{\mu_{\text{eff.}}}\,\rho T\,, \tag{13}$$

then we have for $\psi/kT > 0$,

$$\frac{1}{\mu_{\text{eff.}}} = \eta_{\text{heavy}} + \eta_e\left(1 + \frac{2.15\cdot 10^7}{W^{3/2}} + \dots\right), \tag{14}$$

while for $\psi/kT < 0$,

$$\frac{1}{\mu_{\text{eff.}}} = \eta_{\text{heavy}} + \tfrac{2}{5}\eta_e\left(\frac{-\psi}{kT}\right)\left[1 + \frac{3.9}{\left(\frac{\psi}{kT}\right)^2} - \frac{7.7}{\left(\frac{\psi}{kT}\right)^4} - \frac{9.7}{\left(\frac{\psi}{kT}\right)^6}\dots\right]. \tag{15}$$

For the Fermi energy, $\zeta(= -\psi)$, in terms of ρ, η_e, T, we may write for $\psi/kT > 0$,

$$e^{\psi/kT} = 8.25\cdot 10^{-9}W^{3/2} - 0.354 \tag{16}$$

or

$$\frac{1}{W^{3/2}} = \frac{\rho\eta_e}{T^{3/2}} = 8.25\cdot 10^{-9}e^{-\psi/kT}(1 - 0.354e^{-\psi/kT})\,,$$

for

$$\frac{\psi}{kT} = 0\,;$$

an exact evaluation gives

$$\frac{1}{W^{3/2}} = 6.33\cdot 10^{-9}\,, \tag{17}$$

while for $\psi/kT < 0$ we get

$$\frac{-\psi}{kT} = \frac{2.98\cdot 10^5}{W} - 2.76\cdot 10^{-6}W \tag{18}$$

or

$$\frac{1}{W^{3/2}} = 6.22\cdot 10^{-9}\left(\frac{-\psi}{kT}\right)^{3/2}\left[1 + \frac{1.23}{\left(\frac{\psi}{kT}\right)^2}\right].$$

IV. THE INTEGRATION OF THE STAR EQUATIONS FOR THE SURFACE REGION OF WHITE DWARF STARS

Since the radiation pressure is negligible compared to the gas pressure in a white dwarf star, the two principal star equations become for the surface region

$$\frac{dp_G}{dr} = -\frac{GM}{R^2}\rho \tag{19}$$

$$\frac{dT}{dr} = -\frac{3K\rho L}{16\pi acR^2T^3}, \tag{20}$$

where all the symbols have their usual significance.[9] In equations (18) and (19) we have set $M_r = M$ = total mass, and $r = R$ = total radius; this, as it turns out, is completely justified, since the envelope extends only a small distance inward. We have also taken $L_r = L$ = total luminosity; this assumption, too, is excellent, if the stellar matter is well mixed up. The possibility of all the energy being produced in the envelope will be discussed in section VI. The total opacity K is related to K_R and K_C by

$$\frac{1}{K} = \frac{1}{K_R} + \frac{1}{K_C}. \tag{21}$$

We performed a careful numerical integration[17] of equations (19) and (20) for Sirius B with the observed values of M, L, R, namely,

$$\begin{aligned} M &= M_{\text{obs.}} = 1.95\cdot 10^{33}\ \text{gm}, \\ L &= L_{\text{obs.}} = 9.72\cdot 10^{30}\ \text{ergs/sec}, \\ R &= R_{\text{obs.}} = 1.56\cdot 10^{9}\ \text{cm}, \end{aligned}$$

and with $X_R = 1$; the quantity r was taken as the independent variable, and p_G and T as the dependent variables. Account was taken of the change of functional form of p_G in passing from the nondegenerate to the degenerate part of the envelope (cf. sec. III). The expressions for radiative and conductive opacity given by equations (2), (4), (5), (7), and (8) were used together with Figures 1 and 2 and with values of τ taken from Table 1 of reference 6. The quantity, as

[17] Cf. *Smithsonian Tables*, chapter on "Numerical Integration of Differential Equations."

given in equations (16), (17), (18), was a particularly useful parameter. The results of this calculation are given in Table 2.

Now it is desirable to obtain information concerning the temperature-density distribution for quite different values of M, L, R, X_R. We have, therefore, derived approximate analytical expressions, which can readily be applied and which have been checked by comparison with the above numerical integration. We combine equations (19) and (20) and get

TABLE 2

SURFACE REGION OF SIRIUS B WITH $R=R_{\text{obs.}}$, $M=M_{\text{obs.}}$, $L=L_{\text{obs.}}$

$(R-r)$ $(\cdot 10^{-6})$	ρ $(\cdot 10^{-2})$	T $(\cdot 10^{-6})$	$3I$	τ	ψ/kT	η_e
0	0	0				
2	$9\cdot 10^{-4}$	0.9		1.4	+5.4	0.33
4	$2\cdot 10^{-2}$	1.9	10	2.5	+3.7	0.30
6	0.15	3.2	8.7	4.5	+2.5	0.27
8	0.70	4.6	7.8	8.0	+1.7	0.24
10	1.6	5.5	7.2	14	+1.1	0.23
12	3.0	6.2	6.7	20	+0.6	0.21
14	4.4	6.8	6.5	31	+0.2	0.24
16	5.1	7.2	5.9	38	−0.1	0.25
18	6.9	7.6	5.5	52	−0.6	0.26
20	8.4	8.0	5.1	75	−1.2	0.28
22	14	8.3	5.4	103	−1.8	0.34
24	17	8.6	5.5	115	−2.2	0.35
26	20	8.8	5.5	120	−2.4	0.36
28	23	9.0	5.6	135	−2.7	0.36
30	27	9.2	5.7	145	−3.1	0.37
32	31	9.4	5.8	152	−3.3	0.37
34	36	9.5	5.8	160	−3.4	0.37
36	42	9.6	5.8	166	−3.7	0.37
38	47	9.7	5.9	172	−3.9	0.37
40	53	9.8	5.9	177	−4.2	0.38
46	66	9.9	5.9	189	−5.7	0.38
52	86	10.0	5.9	195	−6.3	0.38
58	107	10.1	5.9	196	−7.7	0.38

$$\frac{d\left(\frac{\rho T}{\mu_{\text{eff.}}}\right)}{dT}=\frac{16\pi acGMT^3}{3\Re KL}\,. \tag{22}$$

This differential equation does not contain the radius of the star. Thus, the relation between temperature and density is independent of R, and the same is true of the temperature at the boundary be-

tween the strongly degenerate core and the nondegenerate surface. This is rather important because of the considerable uncertainty of the observational values of the radii of the white dwarfs. Then equation (22) has to be integrated subject to the condition that $T = 0$ when $\rho = 0$. This can be done directly, provided we make the assumption that

$$\bar{K} = \frac{\int_0^{p_G} K dp_G}{p_G}$$

can be taken equal to K at the point in question.[18] We get

$$K\rho = \mu_{\text{eff.}} T^3 \left[\frac{4\pi acGM}{3\Re L}\right]. \tag{23}$$

To obtain T and ρ as functions of r, we have to insert this expression for $K\rho$ in equation (20) and get

$$\frac{dT}{dr} = -\frac{1}{4}\frac{\mu_{\text{eff.}} GM}{\Re R^2}, \tag{24}$$

with the condition $T = 0$ at $r = R$; equation (24) may be integrated to

$$T = \frac{1}{4}\frac{GM}{\Re R^2}\bar{\mu}_{\text{eff.}} h\,; \tag{25}$$

h is the distance inward from the surface of the white dwarf and

$$\bar{\mu}_{\text{eff.}} = \frac{\int_0^h \mu_{\text{eff.}} dr}{h}.$$

The latter quantity may be estimated fairly readily in the nondegenerate region and at the beginning of the degenerate region; farther in, equation (25) loses its usefulness. In any case, a change of radius of the star will simply change the depth h at which a certain temperature T occurs, in the ratio of R^2.

[18] For the justification of this assumption in the nondegenerate region see Strömgren;[7] as degeneracy sets in, appropriate corrections are easily made.

The approximate formulae (23) and (25) were employed to compute T and h, taking the same values for M, L, R (observed values for Sirius B), X_R, and ρ as in the numerical integration. The agreement with the accurate results given in Table 2 was excellent; for $\psi/kT > -3$ the discrepancy was never more than 10 per cent.[19] For $\psi/kT < -3$ the discrepancy becomes worse, so that equations (23) and (25) will not be applied for stronger degeneracy than that given by $\psi/kT = -3$.

For the purposes of the next section we calculated values of ρ, T, and h for Sirius B and 40 Eridani B under different assumptions as to the radius, luminosity, and chemical composition. These assumptions were:

(*a*) Sirius B:

$M = M_{\text{obs.}}$ $L = L_{\text{obs.}}$ $R = R_{\text{obs.}}$ $X_R = 1$

(*b*) Sirius B:

$M = M_{\text{obs.}}$ $L = L_{\text{obs.}}$ $R = R_{\text{theor.}}$ $X_R = 1$

(*c*) Sirius B:

$M = M_{\text{obs.}}$ $L = L_{\text{obs.}}$ $R = R_{\text{theor.}}$ $X_R = 0$ $X_{He} = 1$ (pure helium star)

(*d*) Sirius B:

$M = M_{\text{obs.}}$ $L = 2L_{\text{obs.}}$ $R = R_{\text{theor.}}$ $X_R = 1$

(*e*) 40 Eridani B:

$M = M_{\text{obs.}}$ $L = L_{\text{obs.}}$ $R = R_{\text{theor.}} \approx R_{\text{obs.}}$ $X_R = 1$

We obtain the results given in Table 3.

V. THE INTERIOR OF WHITE DWARF STARS

In order to obtain an exact solution for the interior of the white dwarfs it would be necessary to integrate simultaneously the entire

[19] In the region $\psi/kT > -3$ the results are quite sensitive to K_R; it was therefore of interest to compare our values of K_R with those given by P. M. Morse's precise calculations (*Ap. J.*, **92**, 27, 1940). It turns out that the agreement is remarkably good, to within a few per cent. I am indebted to Professor Morse for the opportunity to read his paper prior to publication.

set of star equations, taking account of the variation in M_r, L_r, and r (which was not done in the surface region). The equations are[9]

$$\frac{dp_G}{dr} = -\frac{GM_r}{r^2}\rho, \qquad \beta \approx 1 \tag{26}$$

$$\frac{dM_r}{dr} = 4\pi\rho r^2, \tag{27}$$

$$\frac{dT}{dr} = -\frac{3}{16\pi ac}\frac{K\rho}{T^3}\frac{L_r}{r^2}, \tag{28}$$

$$\frac{dL_r}{dr} = 4\pi\rho r^2\epsilon. \tag{29}$$

The energy production per gram-second, ϵ, could be assumed to be given by some thermonuclear reaction such as the proton-proton reaction or the carbon-cycle, and one would adjust the hydrogen

TABLE 3

	h	ρ	T
(a)	$1.4\cdot10^7$	$4.5\cdot10^2$	$6.9\cdot10^6$
	$2.9\cdot10^7$	$2.5\cdot10^3$	$9.1\cdot10^6$
	$4.3\cdot10^7$	$6.0\cdot10^3$	$9.8\cdot10^6$
	$5.8\cdot10^7$	$10.7\cdot10^3$	$10.1\cdot10^6$
(b)	$1.9\cdot10^6$	$4.5\cdot10^2$	$6.9\cdot10^6$
	$3.8\cdot10^6$	$2.5\cdot10^3$	$9.1\cdot10^6$
	$5.7\cdot10^6$	$6.0\cdot10^3$	$9.8\cdot10^6$
	$7.6\cdot10^6$	$10.7\cdot10^3$	$10.1\cdot10^6$
(c)	$3.8\cdot10^6$	$5\cdot10^2$	$5\cdot10^6$
(d)	$7.6\cdot10^6$	$8\cdot10^3$	$11\cdot10^6$
(e)	$1.5\cdot10^7$	$1.4\cdot10^2$	$7\cdot10^6$
	$3.0\cdot10^7$	$2.0\cdot10^3$	$11\cdot10^6$

NOTE: The observed masses, luminosities, and radii were taken from a communication of Professor Chandrasekhar (cf. also G. P. Kuiper, *Ap. J.*, **88**, 472, 1938). The theoretical values were taken from the theory of degenerate configurations (cf. sec. V).

content X_H and the amount of Russell mixture X_R (determined by the helium content) so that the two boundary conditions, $M_r = 0$, $L_r = 0$ at $r = 0$, would be satisfied simultaneously. This would be

tedious and would presumably depend sensitively on the correct choice of ϵ. However, an analysis of Table 2 shows that at a very small distance inward from the surface of the star strong degeneracy is setting in rapidly and the temperature is rising very slowly. The situation will be similar for 40 Eridani B, although not as favorable. This permits us to make a great simplification as a first approximation by neglecting the temperature-dependent terms in the gas pressure, p_G, and to find the temperature distribution without making any explicit assumptions regarding the source of energy. The effects of the temperature may then be taken into account as a perturbation, and the initially assumed mass, radius, and density distribution thereby corrected. It will turn out that this procedure is excellent for both Sirius B and 40 Eridani B.

Now it is well known that when the temperature is neglected the equation of state of a degenerate gas of electrons[5] (valid for both relativistic and nonrelativistic degeneracy) is given by

$$p_G^{\text{elec.}} = 6.01 \cdot 10^{22} f(s) \, , \tag{30}$$

where

$$f(s) = s(2s^2 - 3)(s^2 + 1)^{1/2} + 3 \sinh^{-1} s \, ,$$

with

$$s = 1.01 \cdot 10^{-2}(\rho\eta_e)^{1/3} \, .$$

If we then consider equations (26) and (27) together and introduce the notation

$$\begin{aligned} r &= \alpha\omega \; ; & y &= y_c\phi \\ \alpha &= 7.71 \cdot 10^8 \frac{\eta_e}{y_c} \; ; & y^2 &= s^2 + 1 \, , \end{aligned} \tag{31}$$

where y_c is the central value of y, it is found that these two equations combine into the single one

$$\frac{1}{\omega^2}\frac{d}{d\omega}\left(\omega^2 \frac{d\phi}{d\omega}\right) = -\left(\phi^2 - \frac{1}{y_c^2}\right)^{3/2} \tag{32}$$

subject to the boundary conditions $\phi = 1$, $d\phi/d\omega = 0$ at $\omega = 0$. In terms of ϕ, ω we can write $(\rho\eta_e)$, M_r as follows:

$$(\rho\eta_e) = 9.82 \cdot 10^5 y_c^3 \left(\phi^2 - \frac{1}{y_c^3}\right)^{3/2}, \tag{33}$$

$$\frac{M_r}{\eta_e^2} = -5.70 \cdot 10^{33} \omega^2 \frac{d\phi}{d\omega}. \tag{34}$$

Assuming a set of values for $1/y_c^2$ (really the central density), Chandrasekhar[5] has numerically integrated equation (32) from $\omega = 0$ to $\omega = \omega_1$ (corresponding to $s = 0$); he has also computed tables for $(\rho\eta_e)$ and (M_r/η_e^2) as functions of ω. If η_e is known, a given total mass $M(\omega_1)$ uniquely determines ω_1, and therefore y_c, and thus the radius and density distribution.

Let us first consider Sirius B. The mass of the star is $1.95 \cdot 10^{33}$ and is known directly and very accurately.[20] The temperature in the interior must be at least ten million degrees, because this value is already reached at the boundary of the degenerate core.[21] At such a temperature the proton-proton reaction alone is so probable that only the assumption of a very small hydrogen content in the interior (fraction of a per cent) will not lead to a greater energy evolution than is observed. In order to compute η_e it may, therefore, be assumed that hydrogen is absent; and because of the strong pressure ionization it follows that $\eta_e = 0.50$, no matter what relative amounts of helium and Russell mixture are present. This immediately fixes the constants as

$$\frac{1}{y_c^2} = 0.15 ; \qquad a = 1.5 \cdot 10^8 ; \qquad R = 5.7 \cdot 10^8 \text{ cm} . \tag{35}$$

the mean density $\bar{\rho} = 2.5 \cdot 10^6$; the central density $\rho_c = 2.8 \cdot 10^7$. The radius thus turns out to be much smaller than even the lower one given by Kuiper[22] ($R = 1.36 \cdot 10^9$ cm), but in view of the results

[20] G. P. Kuiper, private communication.

[21] Cf. Table 3. This is true if there is an appreciable amount of heavy elements; otherwise, the boundary temperature is at least five million degrees, which would also require a small hydrogen content...

[22] Cf. end of sec. IV; his value is smaller than Chandrasekhar's.

given below (namely, that the temperature changes only slightly this initial density distribution) and the accuracy claimed for the mass determination, it seems necessary to assume the smaller theoretical radius in our calculation.[23]

Knowing the density distribution, we may now find the temperature distribution by integrating numerically equation (28), subject to the boundary conditions as given by Table 3, i.e.,

ω	T
3.88	$6.9 \cdot 10^6$
3.87	$9.1 \cdot 10^6$
3.86	$9.8 \cdot 10^6$
3.85	$10.1 \cdot 10^6$

For the purposes of the integration some assumption must be made regarding the variation of $(L_r/L)_\omega$ throughout the star; the resulting temperature distribution, together with the density distribution, must then yield an (L_r/L) distribution which is consistent with the initially assumed one. Such a self-consistent distribution can be found only by trial and error.

Our procedure follows. We have

$$\left(\frac{L_r}{L}\right)_\omega = \frac{\int_0^r \rho r^2 \epsilon dr}{\int_0^R \rho r^2 \epsilon dr} . \tag{36}$$

Now, if ϵ is taken to be of the form $\epsilon \approx \rho f(T)$ (which is certainly true for any type of thermonuclear process), then

$$\left(\frac{L_r}{L}\right)_\omega = \frac{\int_0^r \rho^2 f(T) r^2 dr}{\int_0^R \rho^2 f(T) r^2 dr} . \tag{37}$$

It turns out that in the actual self-consistent distribution in Sirius B the temperature changes very slowly in the central region of the star, where most of the energy production takes place. In this case we

[23] As far as the temperature distribution is concerned, the use of the larger radius would have very little effect.

may consider $f(T)$ as slowly varying (in spite of a possible strong dependence of f on T and take it outside the integral sign, thus obtaining

$$\left(\frac{L_r}{L}\right)_\omega = \frac{\int_0^r \rho^2 r^2 dr}{\int_0^R \rho^2 r^2 dr} = \frac{\int_0^\omega \rho^2 \omega^2 d\omega}{\int_0^{\omega_1} \rho^2 \omega^2 d\omega}. \tag{38}$$

For 40 Eridani B, the variation of the temperature is not so slow. In this case we may approximate $f(T)$ by a power of ρ, say, ρ^m. The value of m will depend on the type of energy process; if thermonuclear reactions are considered, it will be higher for a reaction such as the carbon-cycle than for the proton-proton reaction. It will also depend on the variation of temperature with density and is determined by the condition of self-consistency. Then we have[24]

$$\left(\frac{L_r}{L}\right)_\omega = \frac{\int_0^r \rho^{2+m} r^2 dr}{\int_0^R \rho^{2+m} r^2 dr} = \frac{\int_0^\omega \rho^{2+m} \omega^2 d\omega}{\int_0^{\omega_1} \rho^{2+m} \omega^2 d\omega}. \tag{39}$$

For Sirius B a set of values for $(L_r/L)_\omega$ was found from equation (37). We then integrated equation (28), taking account of the variation of η_e in the outer region.[25] It is found that the variation of T with ρ is about as $T \sim \rho^{1/75}$. Even if the carbon-cycle is assumed, $f(T)$ varies only as $T^{20} \sim \rho^{1/4}$; for the proton combination the variation would be only as $T^4 \sim \rho^{1/20}$. Thus, the choice $m = 0$ is obviously a better one than, say, $m = 1$. This is so because $m = 1$ gives just about the same (T, ρ) dependence as $m = 0$. To test the sensitivity of the results to inaccuracies in the luminosity, the temperature distribution was calculated on the assumption $L = 2L_{\text{obs.}}$, $X_R = 1$, and it is found that $T_c = 18.4 \cdot 10^{6\circ}$ C. An uncertainty in the luminosity of a factor of 10 would produce an uncertainty in the central temperature of a factor of 2. For Kuiper's value of the luminosity, the

[24] To get an even better approximation, one could break up the range $(0, \omega_1)$ into subregions, associating a slightly different (nonintegral) value of m with each subregion; this was not done in our calculation for reasons which will become apparent below.

[25] The quantity η_e is different from 0.50 only for the first few points, so that it is justified to take $\eta_e = 0.50$ for the star as a whole.

central temperature of Sirius B is thus $15.2 \cdot 10^{6\circ}$ C, and is fairly constant over a larger part of the degenerate core. The complete temperature distribution is given in Table 4.

For 40 Eridani B, the mass was taken as $7.95 \cdot 10^{32}$ gm, i.e., $0.40\, M_{\odot}$, instead of Kuiper's value $0.45\, M_{\odot}$; this is much more

TABLE 4

SIRIUS B (T IN UNITS OF 10^6)

ω	(ρ/ρ_c)	$(L_r/L)_\omega$	T
0.0	1.000	0.000	15.20
0.2	0.980	0.003	15.20
0.4	0.930	0.022	15.20
0.6	0.850	0.069	15.20
0.8	0.752	0.138	15.20
1.0	0.644	0.228	15.18
1.2	0.537	0.344	15.16
1.4	0.436	0.470	15.14
1.6	0.346	0.587	15.11
1.8	0.268	0.690	15.07
2.0	0.203	0.782	15.03
2.2	0.151	0.855	14.98
2.4	0.109	0.920	14.91
2.6	0.0765	0.958	14.83
2.8	0.0518	0.980	14.73
3.0	0.0333	0.986	14.60
3.2	0.0199	0.990	14.45
3.4	0.0104	1.000	14.10
3.6	0.00429	1.000	13.60
3.7	0.00220	1.000	12.90
3.8	0.00096	1.000	11.50
3.85	0.00036	1.000	10.10
3.86	0.00021	1.000	9.80
3.87	0.00009	1.000	9.10
3.88	0.00002	1.000	6.90

convenient for the purposes of Chandrasekhar's tables and is permissible in view of the considerable uncertainty of the mass determination; this was not the case for Sirius B. Then the theoretical radius of 40 Eridani B is $1.05 \cdot 10^9$, as compared with Kuiper's value of $1.25 \cdot 10^9$, and

$$\omega_1 = 3.60 \; ; \qquad \frac{1}{y_c^2} = 0.6 \; ; \qquad a = 3.0 \cdot 10^8 \; ; \tag{40}$$

$$\bar{\rho} = 1.67 \cdot 10^5 \; ; \qquad \rho_c = 1.15 \cdot 10^6 \; .$$

We have integrated equation (28) for 40 Eridani B in the same manner as for Sirius B with $L = L_{\text{obs.}} = 2.1 \cdot 10^{31}$, $X_R = 1$, subject to the boundary conditions

ω	T
3.55	$7 \cdot 10^6$
3.50	$11 \cdot 10^6$

Table 5 contains the resulting temperature distribution corresponding to a choice of $m = 0$ and $m = 1$ for the $(L/L_r)_\omega$ distribution.

TABLE 5

40 ERIDANI B (T IN UNITS OF 10^6)

ω	$\left(\frac{\rho}{\rho_c}\right)$	$\left(\frac{L_r}{L}\right)_\omega$ for $m=0$	$\left(\frac{L_r}{L}\right)_\omega$ for $m=1$	T for $m=0$	T for $m=1$
0.0	1.000	0.000	0.000	30.2	31.0
0.2	0.987	0.00414	0.008	30.1	30.9
0.4	0.951	0.0231	0.042	30.0	30.8
0.6	0.893	0.0729	0.128	29.9	30.6
0.8	0.818	0.154	0.256	29.7	30.3
1.0	0.731	0.267	0.406	29.5	30.0
1.2	0.637	0.391	0.566	29.2	29.6
1.4	0.542	0.525	0.710	28.9	29.2
1.6	0.450	0.654	0.820	28.6	28.8
1.8	0.364	0.760	0.902	28.3	28.4
2.0	0.286	0.849	0.940	27.9	27.9
2.2	0.218	0.909	0.975	27.4	27.4
2.4	0.160	0.944	0.990	26.7	26.7
2.6	0.112	0.978	0.994	25.9	25.9
2.8	0.0732	0.987	0.997	25.0	25.0
3.0	0.0434	0.997	1.000	24.0	24.0
3.2	0.0216	0.999	1.000	21.8	21.8
3.3	0.0131	1.000	1.000	21.0	21.0
3.35	0.0100	1.000	1.000	19.5	19.5
3.40	0.0072	1.000	1.000	17.5	17.5
3.45	0.0042	1.000	1.000	14.5	14.5
3.50	0.0017	1.000	1.000	11.0	11.0
3.55	0.00012	1.000	1.000	7.0	7.0

tion. Both solutions give a dependence of T on ρ of the form $T \sim \rho^{1/15}$ and differ very little. Since

$$f(T) \sim \rho^{1/5} \tag{41}$$

for the proton-proton reaction and

$$f(T) \sim \rho$$

for the carbon-cycle, the choice $m = 0$ would be better for the first reaction and $m = 1$ for the second. Not only does the internal temperature distribution vary more in 40 Eridani B than in Sirius B, but a higher central temperature is attained—about thirty million degrees. This is due to the smaller mass densities in 40 Eridani B.

Before any conclusions can be drawn regarding the hydrogen content, etc., from the above temperature and density distributions, the effect of the temperature in changing the initially assumed density distribution must be calculated. This may be done by treating the temperature as a perturbation in equations (26) and (27). Combining equations (26) and (27), we get

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{dp_G}{dr}\right) - 3\frac{dp_G}{dr}\frac{d\log s}{dr} = -Cs^6\,, \tag{42}$$

where we have taken (eq. [30]) $\rho = Bs^3$, $C = 4\pi GB^2$. Neglecting the heavy-particle pressure,[26] we write[5]

$$p_G = Af(s)\left[1 + \frac{as(s^2+1)^{1/2}}{f(s)} + \ldots\right]; \tag{43}$$

the temperature perturbation is $a = 4\pi^2k^2T^2/m^2c^4$, $A = 6.01\cdot 10^{22}$, and m is the electron mass. Since the degenerate core of a white dwarf is practically isothermal, we take a as constant throughout the star; e.g., for Sirius B, $T = 15\cdot 10^{6\circ}$ C. We may, therefore, expand p_G about the point $T = 0$ as follows:

$$p_G = Af(s_0) + Aa\left[s_0(s_0^2+1)^{1/2} + f'(s_0)\left(\frac{\partial s}{\partial a}\right)_0\right]. \tag{44}$$

Also

$$s^6 = s_0^6 + 6s_0^5\left(\frac{\partial s}{\partial a}\right)_0 a\,,$$

$$\log s = \log s_0 + \frac{a}{s_0}\left(\frac{\partial s}{\partial a}\right)_0 .$$

Substituting these expressions into equation (42) and making use of the unperturbed equation

$$\frac{1}{r^2}\frac{d}{dr}\left[r^2\frac{d}{dr}Af(s_0)\right] - 3\frac{d}{dr}\left[Af(s_0)\right]\cdot\frac{d\log s_0}{dr} = -Cs_0^6 .$$

[26] This is justified, as can be seen by finding the value of ρ for which the heavy-particle pressure (necessarily temperature dependent) contributes more than the temperature-dependent part of the electron gas pressure. It turns out that the heavy particles contribute in the region where the temperature perturbation is very slight. At any rate, the order of magnitude will be given by our calculation.

We get the following equation for $z = (\partial s/\partial a)_0$:

$$\left.\begin{aligned}&\frac{1}{r^2}\frac{d}{dr}\left[r^2\frac{d}{dr}\{s_0(s_0^2+1)^{1/2}+f'(s_0)z\}\right]-\frac{3df(s_0)}{dr}\frac{d}{dr}\left(\frac{z}{s_0}\right)\\&\quad-3\frac{d}{dr}\{s_0(s_0^2+1)^{1/2}+f'(s_0)z\}\cdot\frac{d}{dr}\log s_0=-6s_0^5\frac{C}{A}z\,.\end{aligned}\right\}\quad(45)$$

If the exact expression is used for $f(s_0)$, equation (45) is somewhat cumbersome to work with; however, it is consistent with our approximation to take

$$f(s_0) = \tfrac{8}{5}s_0^5 \quad \text{for nonrelativistic degeneracy} \quad \left(s_0 < \frac{5}{4}\right),$$

$$f(s_0) = 2s_0^4 \quad \text{for relativistic degeneracy} \quad \left(s_0 > \frac{5}{4}\right).$$

The boundary between the two regions is given by $s_0 = \frac{5}{4}$, corresponding to a mass density $\rho = 3.8 \cdot 10^6$ (if η_e is taken as 0.50).

With the indicated simplifications, equation (45) becomes, in the region of relativistic degeneracy,

$$\left.\begin{aligned}&\frac{d^2z}{dr^2}+\frac{2}{r}\frac{dz}{dr}+\left[\frac{3}{s_0}\frac{d^2s_0}{dr^2}+\frac{6}{s_0r}\frac{ds_0}{dr}+\frac{3Cs_0^2}{4A}\right]z\\&\qquad=\left[\frac{1}{2s_0^3}\left(\frac{ds_0}{dr}\right)^2-\frac{1}{4s_0^2}\frac{d^2s_0}{dr^2}-\frac{1}{2rs_0^2}\frac{ds_0}{dr}\right].\end{aligned}\right\}\quad(46)$$

In the region of nonrelativistic degeneracy, we get

$$\left.\begin{aligned}&\frac{d^2z}{dr^2}+\left(\frac{2}{r}+\frac{2}{s_0}\frac{ds_0}{dr}\right)\frac{dz}{dr}+\left[\frac{3}{s_0^2}\left(\frac{ds_0}{dr}\right)^2+\frac{8}{rs_0}\frac{ds_0}{dr}+\frac{3Cs_0}{4A}+\frac{4}{s_0}\frac{d^2s_0}{dr^2}\right]z\\&\qquad=\left[\frac{3}{8s_0^5}\left(\frac{ds_0}{dr}\right)^2-\frac{1}{8s_0^4}\frac{d^2s_0}{dr^2}-\frac{1}{4rs_0^4}\frac{ds_0}{dr}\right].\end{aligned}\right\}\quad(47)$$

If we express s_0 in terms of Chandrasekhar's function, i.e.,

$$s_0^2 + 1 = y_c^2\phi^2\,; \quad \frac{ds_0}{dr} = \frac{y_c^2}{\alpha s_0}\left(\phi\frac{d\phi}{d\omega}\right),$$

equation (46) can be written

$$\frac{d^2\bar{z}}{dr^2} + F(r)\bar{z} = G(r)\,, \qquad (48)$$

$$\bar{z} = rz\,; \qquad r = \alpha\omega\,,$$

where

$$F(r) = -\frac{3}{s_0^2}\left(\frac{ds_0}{dr}\right)^2 + \frac{3y_c^2}{s_0^2\alpha^2}\left(\frac{d\phi}{d\omega}\right)^2 - \frac{3\phi s_0}{\alpha^2 y_c} + \frac{3Cs_0^2}{4A},$$

and

$$G(r) = \frac{3r}{4s_0^3}\left(\frac{ds_0}{dr}\right)^2 - \frac{ry_c^2}{4s_0^3\alpha^2}\left(\frac{d\phi}{d\omega}\right)^2 + \frac{\phi r}{4\alpha^2 y_c};$$

and equation (47) can be written

$$\frac{d^2z}{dr^2} + B(r)\frac{dz}{dr} + C(r)z = D(r), \tag{49}$$

where

$$B(r) = \frac{2}{s_0}\left(\frac{ds_0}{dr}\right) + \frac{2}{r},$$

$$C(r) = \frac{4y_c^2}{\alpha^2 s_0^2}\left(\frac{d\phi}{d\omega}\right)^2 - \frac{4s_0\phi}{\alpha^2 y_c} - \frac{1}{s_0^2}\left(\frac{ds_0}{dr}\right)^2 + \frac{3Cs_0}{4A},$$

and

$$D(r) = \frac{1}{2s_0^5}\left(\frac{ds_0}{dr}\right)^2 - \frac{y_c^2}{8s_0^5\alpha^2}\left(\frac{d\phi}{d\omega}\right)^2 + \frac{\phi}{8s_0^5\alpha^2 y_c}.$$

We have integrated equation (48) for Sirius B, taking $1/y_c^2 = 0.2$,[27] subject to the boundary conditions $\bar{z} = 0$, $d\bar{z}/dr = 0$ at $\omega = 0$. The integration was carried up to the point $\omega = 1.9$, where $s_0 = \frac{5}{4}$. At $\omega = 1.9$,

$$\bar{z} = 20.9 \cdot 10^6, \qquad \frac{d\bar{z}}{dr} = 6.08 \cdot 10^{15};$$

this is equivalent to

$$\left.\begin{aligned} z &= 6.37 \cdot 10^{-2} \\ \frac{dz}{dr} &= 4.43 \cdot 10^{-10} \end{aligned}\right\} \quad \text{at} \quad \omega = 1.9.$$

Equation (49) was then integrated, subject to these boundary conditions, up to the point $\omega = 3.67$.[28] From this point to the boundary of the star ($\omega_1 = 3.727$), equation (49) can no longer be used.

[27] This corresponds to a mass $M = 0.88M_\odot$ and $\alpha = 1.732 \cdot 10^8$; this choice was convenient and will give a close upper bound to the effect of the temperature perturbation.

[28] The author is indebted to Mr. Robert Mann of the college of arts and sciences, University of Rochester, for the calculation of the functions F, G, A, B, C, D.

With a knowledge of the z distribution, the modification in mass has been estimated as follows: Let $\rho = \rho_0 + \rho_1$, where ρ_0 is the density distribution corresponding to $T = 0$ and is given by Chandrasekhar's tables.[5] The quantity ρ_1 is the perturbation term due to the finite temperature. Then $\rho_1 = 3Bs_0^2za$, and the change in mass ΔM is

$$\Delta M = \frac{4\pi}{3}\int_0^R r^2\rho_1 dr = 4\pi Ba\int_0^R r^2 s_0^2 z dr = \Delta_1 M + \Delta_2 M ,$$

where

$$\Delta_1 M = 4\pi Ba\alpha^3\int_0^{3.67} \omega^2 s_0^2 z d\omega = 3.8 \cdot 10^{-5} M ,$$

for

$$T = 1.5 \cdot 10^{7\circ} C$$

$\Delta_2 M$ is the uncertainty introduced in the region $\omega = 3.67$ to $\omega_1 = 3.73$, and if we take the total mass in this region then at most $\Delta_2 M = 1.5 \cdot 10^{-4} M$. Hence, we may say

$$3.8 \cdot 10^{-5} < \frac{\Delta M}{M} < 1.9 \cdot 10^{-4} .$$

For Sirius B, whose mass is just about equal to the sun's mass, it is clear that a temperature perturbation of $15 \cdot 10^{6\circ}$ C—which corresponds to the initial temperature—will lead to a ΔM about one-hundredth of 1 per cent of the original mass. The theoretical radius must remain, and the initial temperature-density distribution may be used to gain an estimate of the concentration of reactants which must be present if the observed luminosity is assumed to be due to the occurrence of nuclear processes involving these reactants.

A much more approximate calculation was performed for 40 Eridani B, in order to estimate the effect of the temperature in changing the unperturbed mass-density distribution. We took $1/y_c^2 = 0.6$, corresponding to a mass $M = 0.04\ M_\odot$, and for T the value $30 \cdot 10^{6\circ}$ C throughout. It turns out that the change in mass is negligible again, i.e., $\Delta M = 3 \cdot 10^{-3} M$.

We may now compute the hydrogen content of Sirius B and of 40 Eridani B if we make an assumption regarding the source of energy. From equation (29) it follows that

$$L = 4\pi\int_0^R \rho\epsilon r^2 dr . \tag{50}$$

Suppose we were to assume that the carbon-cycle is responsible for the energy production in the white dwarfs (that is, that the concen-

tration of carbon and nitrogen is the same as in main-sequence stars, say, $\frac{1}{2}$ of 1 per cent by weight),[3a] then we should have

$$\epsilon = 2.95 \cdot 10^{21} X_H \rho \tau^2 e^{-\tau} \; ; \qquad \tau = \frac{152}{T^{1/3}} \, ,$$

with T measured in millions of degrees.[1] Inserting this expression into equation (50), using the temperature-density distributions found for Sirius B and 40 Eridani B,[29] and taking $L = L_{\text{obs.}}$ ($9.7 \cdot 10^{30}$ ergs/sec for Sirius B, $2.1 \cdot 10^{31}$ ergs/sec for 40 Eridani B), we get

for Sirius B. . . . . $X_H = 2.2 \cdot 10^{-8}$;
for 40 Eridani B. . . . . . $X_H = 3.6 \cdot 10^{-7}$.

Since it might be true that carbon and nitrogen are absent in the white dwarfs,[30] we have also computed the hydrogen content in case the only reactants are the hydrogen nuclei themselves (i.e., proton-proton reaction).[2] Then

$$\epsilon = 4 \cdot 10^3 X_H^2 \rho \tau^2 e^{-\tau} \; ; \qquad \tau = \frac{33.8}{T^{1/3}} \, ,$$

with T expressed in millions of degrees, and we get

for Sirius B. . . . . $X_H = 2.4 \cdot 10^{-5}$,
for 40 Eridani B. . . . $X_H = 8.0 \cdot 10^{-5}$.

These values certainly give the maximum hydrogen content under any reasonable assumptions regarding the chemical composition (i.e., an appreciable concentration of heavy elements).

However, we thought it worth while to consider the extreme case of a star consisting of helium[31] alone. This lowers the temperature be-

[29] Cf. Tables 4 and 5.

[30] In a recent paper (*M.N.*, **99**, 595, 1939) Eddington discusses the hydrogen content of the white dwarfs. In addition to the fact that he makes no attempt to calculate the temperature distribution in the white dwarfs, he considers only the carbon-cycle in making his estimates. His statements are therefore less conclusive than ours, especially regarding the radii of the white dwarfs.

[31] In this calculation heavy helium is implied. In view of the work of Alvarez and Cornog (*Phys. Rev.*, **56**, 379, 1939), it might be thought that He^3 should be investigated as a possible important constituent of the white dwarfs. However, there are almost insurmountable objections against an appreciable abundance of He^3 in the white dwarfs (cf. W. A. Wildhack, *Phys. Rev.*, **57**, 81, 1940).

cause the opacity, both radiative and conductive, is reduced by a factor of 6 ($Z^2/A = 1$ instead of 6 as for the Russell mixture). The boundary temperature obtained by the method of section V is $5 \cdot 10^6$ instead of $10 \cdot 10^{6\circ}$ C. In the transition region there is a slight additional effect in the same direction due to the greater density of free electrons η_e. The contribution of the nuclei to the gas pressure is not essential, for the higher η_{heavy} is nearly compensated by the lower temperature. The central temperature for Sirius B is then $7 \cdot 10^6$, i.e., about one-half of the value for the Russell mixture. It is necessary to assume $X_H = 1.5 \cdot 10^{-4}$ in order to give $L = L_{\text{obs}}$. Thus, it was certainly legitimate to take $X_H = 0$ and to set $\eta_e = 0.50$ in deducing the radius from the mass and in calculating the temperature distribution.

VI. CONCLUSIONS

We have found that the hydrogen content for Sirius B and 40 Eridani B must be very low;[32] this would be true for the other white dwarfs as well, in view of their low luminosities and high densities. This result is true, no matter whether the carbon-cycle or the proton-proton reaction is assumed to be responsible for the energy production, and remains true even if the white dwarfs were taken to be pure helium stars. It is most plausible to assume that white dwarfs contain some heavier elements as well, and among them carbon and nitrogen. This would almost certainly follow if the white dwarfs are the last stages in the evolution of main-sequence stars, since all stars in the main sequence (except possibly stars much less massive than the sun)[3] contain a considerable amount of Russell mixture and consume only hydrogen in the course of their evolution.[1] Even if the white dwarfs are not the final stages in the evolution of main-sequence stars, it is likely that carbon and nitrogen are present.

Assuming, then, the same carbon and nitrogen concentration as in main-sequence stars, the hydrogen content of Sirius B will be $2 \cdot 10^{-8}$ and of 40 Eridani B $4 \cdot 10^{-7}$. Such a low hydrogen content would supply the energy, even with the observed low luminosities, only for a

[32] In making estimates of the hydrogen content on the basis of the proton-proton reaction, the numerical factor $4 \cdot 10^3$ has been used. This differs from equation (35) n. 2, by a factor of 10 (cf. B. O. Grönblom, *Phys. Rev.*, **56**, 508, 1939, and R. E. Marshak and H. A. Bethe, *Phys. Rev.*, **56**, 210, 1939); otherwise X_H would be three times larger.

very limited time. The energy evolved in the transformation of hydrogen into helium is 10^{-5} ergs per proton, and therefore, altogether, $E \approx 6 \cdot 10^{23} \cdot 10^{-5} X_H$ ergs/gm. For Sirius B we obtain, since $X_H \approx 2 \cdot 10^{-8}$, $E \approx 1.2 \cdot 10^{11}$ ergs/gm. The luminosity of Sirius B is 0.005 ergs/gm-sec; therefore, the hydrogen would supply the energy for $\approx 1.2 \cdot 10^{11}/0.005$ sec $\approx 8 \cdot 10^5$ years, i.e., only for a very short time. For 40 Eridani B we find about the same time, viz.,

$$1.2 \cdot 10^{19} \times \frac{4 \cdot 10^{-7}}{0.026} \text{ sec} \approx 3 \cdot 10^6 \text{ years} .$$

Since, in general, such short lifetimes would be incompatible with the observed frequency of the white dwarfs, it seems that the present hydrogen content cannot be responsible for the energy production.

There are at least two alternative possibilities for the energy production: (1) The hydrogen concentration in the surface layers may be very much larger than in the interior, as seems to be indicated by observations of the spectra.[33] Then, either the energy production can take place in the surface layers themselves, or the hydrogen could diffuse into the interior and thereby replenish the source of protons. (2) The energy production may be due largely to gravitation.

The first possibility involving the diffusion of hydrogen would probably require too long a time to play a significant role, since diffusion through a degenerate gas is an extremely slow process. The second alternative seems rather plausible, from the assumption that the white dwarfs are produced either by gravitational contraction of main-sequence stars or by the contraction of fragments of giant explosions. The small radii which the white dwarfs possess furnish further support for hypothesis (2).

Thus, it might be supposed that the white dwarfs have evolved from main-sequence stars of the same mass. Then the hydrogen content, say, for a star of solar mass, at the time when gravitational contraction produces as much energy as the thermonuclear processes

[33] Cf. Öhman, *Arkiv. f. Mat. Astron. och Fysik, Stockholm*, **25**B, no. 21, 1937. Professor Kuiper has also kindly informed the writer that there are at least two distinct types of white dwarf spectra: the type which contains no trace of hydrogen, and the type exemplified by Sirius B, which exhibits fairly strong hydrogen lines.

is of the order of 10^{-5}.[34] This is compatible with the value of the hydrogen content obtained above, even with the assumption that only the proton-proton reaction is taking place. However, the time required for the hydrogen to be consumed (through its conversion into helium) from its 35 per cent concentration in a main-sequence star of solar mass to a small fraction of 1 per cent is of the order of 10^{10} years. This long time scale would be difficult to accept if the concept of the expanding universe is correct—according to which the great nebulae were formed about $2 \cdot 10^9$ years ago and the individual stars not much earlier than that.[35] In view of this difficulty, it might be necessary to postulate with Gamow[36] that the white dwarfs have originated in the explosion of main-sequence stars of masses larger than Chandrasekhar's critical mass,[37] which have contracted greatly without degeneracy having set in. Presumably, the hydrogen content at the time of explosion would be small, since stars of large mass have a large luminosity and hence consume their hydrogen much more rapidly than stars of solar mass. The white dwarfs would then arise from the further gravitational contraction of the fragments.

If we grant that thermonuclear processes are absent, the life of a white dwarf (time before it turns into a dark object)will be given by the difference between the gravitational energy which the white dwarf has at present and that which it possesses in a completely degenerate state, namely, $E_G = \gamma GM^2/R \cdot \Delta R/R$; γ is a constant of order of unity, and ΔR is the difference between the radius in the completely degenerate state and the radius at finite temperature. From Chrandrasekhar's tables[38] it is seen that $\Delta R/R \approx \Delta M/M$, where $\Delta M/M$ is essentially the quantity computed in the perturbation calculation (cf. sec. V). Thus for Sirius B, $\Delta M/M \approx 10^{-4}$ and $E_G \approx 5 \cdot 10^{46}$ ergs, while for 40 Eridani B, $\Delta M/M \approx 3 \cdot 10^{-3}$ and

[34] For the sun this would happen when $R \approx 0.8 R_\odot$, $X_H \approx 10^{-5}$ (cf. G. Gamow, *Phys. Rev.*, **55**, 718, 1939); the result is not very sensitive to the mass.

[35] G. Gamow and E. Teller, *Phys. Rev.*, **55**, 654, 1939.

[36] *Phys. Rev.*, **55**, 718, 1939.

[37] Masses larger than $6 \cdot 65\ \eta_e^2 M_\odot$ can never become degenerate (cf. Chandrasekhar).[5]

[38] Chandrasekhar, *op. cit.*, Table 5, p. 427.

$E_G \approx 2 \cdot 10^{47}$ ergs. To be more accurate we should also consider the thermal, radiative, and ionization contributions to the total available energy. It can easily be shown for Sirius B and 40 Eridani B, from the temperature-density distributions which have been calculated, that $E_{\text{rad.}}$, and $E_{\text{ioniz.}}$ are negligible compared to E_{thermal}. Now E_{thermal} is due chiefly to the electrons and is equal to $\int c_V T dV$, where

$$c_v = \frac{\pi^2 k^2}{mc^2} \frac{(s^2 + 1)^{1/2}}{s^2} \frac{T \rho \eta_e}{m_H},$$

with

$$s = 1.01 \cdot 10^{-2} (\rho \eta_e)^{1/3} .$$[39]

We can write

$$E_{\text{thermal}} = 1.7 \cdot 10^6 a^3 y_c \int_0^{\omega_1} T^2 s \phi \omega^2 d\omega .$$

We get

For Sirius B . . . . $E_{\text{thermal}} = 2.5 \cdot 10^{46}$ ergs;
For 40 Eridani B . . . . $E_{\text{thermal}} = 1.5 \cdot 10^{47}$ ergs.

Thus, the available energy is given essentially by the gravitational contribution and is about 10^{47} ergs. If it is assumed that the energy continues to be radiated at the present rate (this is an upper limit; it can be shown that as complete degeneracy is more nearly reached the velocity of contraction diminishes), we have for Sirius B, $L \approx 10^{31}$ ergs/sec, so that the minimum life is 10^{47} ergs/10^{31} ergs/sec $\approx 3 \cdot 10^8$ years. For 40 Eridani B the minimum life is $5 \cdot 10^8$ years. These results are quite consistent with the density of white dwarfs in space and with an age for the universe of $\sim 10^9$ years.

While the above arguments for gravitational contraction as the source of energy in white dwarf stars are admittedly inconclusive, the situation with regard to the predicted radii is much more definite. The agreement found for 40 Eridani B on the assumption of negligible hydrogen content is quite satisfactory, while the theoretical value of the radius of Sirius B disagrees completely with the observed radius—by more than a factor 2. In view of the fact that the theoretical radii seem to be as certain as the theory of degenerate con-

[39] *Ibid.*, p. 394.

figurations (embodying relativistic effects) and the physical principles underlying the computation of the rate of hydrogen combination, it is worth while to inquire whether any reasonable modifications of our assumptions can lead to an increased radius for Sirius B.

In the case of Sirius B, even if it is true that carbon and nitrogen are completely absent, the proton-proton reaction gives an upper limit of $2 \cdot 10^{-5}$ for the hydrogen content. Unless the selection rules for β-disintegration are incorrect (in which case the probability would be decreased by a large factor $\sim 10^6)^2$ the reaction rate cannot be wrong by more than a factor of 10. Then it must follow that $\mu_e = 2$ (since only for hydrogen, in contrast to all other elements[40] under very high pressures, is $\mu_e = 1$) and from the theory of degenerate configurations that $R \approx 0.6 \cdot 10^9$ cm—to be compared with an observed value of $1.4 \cdot 10^9$ cm. It might be thought that because of relativistic degeneracy the electrons will possess sufficient energy to combine with protons and form neutrons; however, a neutron core will not arise, because so little hydrogen is present that the few neutrons which are formed will be captured by other nuclei. The pressure is not high enough in Sirius B for the conversion of heavier nuclei into neutrons; furthermore, the formation of a neutron core would only aggravate the situation. Again, it might be said that the integration of the star equations near the surface, assuming $L_r = L$, was not justified if it is true that the energy is produced in the envelope and that the degenerate core does not contain any energy sources. But this will not help matters; the only consequences will be that the temperature gradient will decrease, degeneracy will set in sooner, and the envelope will still be very small. The discrepancy in radius will remain. The only conclusion we can draw, therefore, is that the observations of the radius of Sirius B are in error.

We are very reluctant to draw such a conclusion, in view of the approximate agreement obtained between two distinct methods of measurement of the radius—by the $(L, T_{\text{eff.}})$ relationship and by the Einstein gravitational red shift. Moreover, G. P. Kuiper has recently re-examined the observational material for Sirius B. He con-

[40] Except for He^3; but the abundance of He^3 must be inappreciable in the white dwarfs.

cludes that it is extremely improbable that the measured radius is in error by a factor of 2. He points out that the relativistic red shift would have to be 80 km/sec, instead of the present 20 km/sec, and the effective surface temperature 25,000° instead of 10,000° (the value indicated by the spectrum), to get agreement with the theoretical radius. He concludes that only "the *ad hoc* assumption of a close, faint companion to Sirius B, which is at the same time, a white dwarf, could bring μ_e to the value 2."[41]

But, on the other hand, the theory seems to be well founded. It is obvious that no agreement can be hoped for with a central temperature of $15 \cdot 10^{6\circ}$ C (Russell mixture). With $T_c \approx 7 \cdot 10^{6\circ}$, as is obtained for a pure helium star and a minute abundance of C and N, the carbon-cycle will be sufficiently slow, but the proton-proton reaction will still be fast. The only possibility would be to make the additional assumption that the theory of the proton-proton reaction is completely wrong—but this would require a far-reaching revision of our present ideas concerning nuclear structure and is not very likely. The dubious validity of rejecting the Gamow-Teller selection rules and returning to the original Fermi version of the β-decay (which as mentioned above would decrease the probability of the proton-proton reaction by a large factor—a factor probably sufficient to permit the presence of an appreciable amount of hydrogen in Sirius B and therefore a larger radius) is attested to by recent experimental results on the β disintegration of light nuclei. These experimental results seem to be more readily explained by a theory of β-decay which is spin dependent rather than by one which is not.[42] In particular, the strikingly large probability of the $He^6 \rightarrow Li^6 + e^-$ reaction[43] is presumed to provide strong confirmation of the Gamow-Teller theory. Since the combination of the two protons to form a deuteron with the subsequent emission of a positron is wholly analogous to $He^6 = He^4 + 2n \rightarrow Li^6 + e^- = He^4 + 1p + 1n + e^-$ (in virtue of the equality of like particle forces), it would follow that Gamow-Teller selection rules should be employed in the computa-

[41] The author is indebted to Professor Kuiper for sending him his paper prior to publication.

[42] Cf. B. O. Grönblom, *op. cit.*, p. 509, and E. P. Wigner, *Phys. Rev.*, **56**, 519, 1939.

[43] Cf. T. Bjerge and K. J. Bröstom, *Nature*, **138**, 400, 1936.

tion of the proton-proton reaction rate in stars. However, there is a bare possibility that the spin of He^6 is one, in which case the Gamow-Teller selection rules would be unnecessary. Whether or not the remaining experimental evidence for the Gamow-Teller rules can be explained by similar modifications in nuclear theory cannot definitely be settled at present. All such modifications would be difficult to reconcile with our present-day knowledge of nuclear forces. The present investigation has at least established almost beyond question that the claim of astrophysics is in direct conflict with the claim of nuclear physics and that there really seems to be no simple explanation of the radius discrepancy for Sirius B.

In conclusion, the author would like to express his gratitude to Professor H. A. Bethe for much valuable help at different stages in the investigation and also for a critical revision of the entire manuscript. He would also like to thank Professors S. Chandrasekhar and G. P. Kuiper for their kindness in supplying necessary astrophysical data and tables.

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May 1940

THE SYSTEMATIC AND ACCIDENTAL ERRORS OF SPECTROSCOPIC PARALLAXES*

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ABSTRACT

I. The stars for which trigonometric and spectroscopic parallaxes are available were divided into eight groups of approximately equal number, in order of the "reduced" spectroscopic parallaxes and of the reduced proper motions, and the mean reduced spectroscopic and trigonometric parallaxes were compared. *The deviations from linear correlation between the two are no larger than can be attributed to the accidental and sampling errors*—both for the biased and impartial groupings and also for four ranges of spectral class among main-sequence stars, for all these together, and for giants. To distinguish between the results of the assumptions of linear correlation between the absolute magnitudes and between the reduced parallaxes would demand several times the observational data now available (1140 stars). For stars differing greatly in absolute magnitude from the general mean (such as Groombridge 1830), these linear correlations are probably invalid; and the data are at present insufficient for a reliable calibration.

II. The *dispersion constants for the reduced parallaxes have been redetermined* by analysis of these mean values (Table 4) and compared with those obtained by our earlier correlation analysis (Table 5). The factor $1/l$, by which the differences of the tabular spectroscopic absolute magnitude from the mean for the spectral subclass must be multiplied in order to obtain an impartial calibration, comes out consistently larger by the latter method.

III. The correlation between the reduced parallaxes and proper motions (which is equivalent to that between the absolute magnitude M and $H = m + 5 \log \mu$) shows that, for groups brighter and fainter than the normal for the spectral subclass, $\Delta M/\Delta H = 0.74$ for main-sequence stars and 0.66 for giants. Comparison of giants and dwarfs gives $\Delta M/\Delta H = 0.69$; of stars in different parts of the main sequence, 0.80.

IV. *Unavoidable deviations* of spectroscopic absolute magnitudes from the true values may arise from *physical causes within* the stars, such as *differences in the relative abundance of hydrogen and heavy elements*, either in the interior or in the atmosphere, and *differences in the contribution of negative hydrogen ions to the general opacity*.

These differences may account for a considerable portion of the apparent "errors" of spectroscopic parallaxes.

V. These effects, combined with a correlation between the masses and the space-velocities of the stars, may cause grouping by H to be systematically different from one by M; but this appears inadequate to account for the systematic differences mentioned above.

VI. Since no conclusive reasons have been found for preferring one of the sets of constants described in (II) to the other, weighted means have been taken, and the remaining dispersion constants have been determined with the aid of these. Results for the *reduced parallaxes* appear in Table 9; and the corresponding constants for the *absolute magnitudes* (derived on the assumption that these have a normal distribution), in Table 10.

The mean-square difference θ between the spectroscopic absolute magnitudes, M_s, as published in *Mt. W. Contr.* No. 511, and the true values is $\pm 0^m48$ for main-sequence stars as a whole and $\pm 0^m44$ for giants. Joy's values, derived from double stars and clusters, are in substantial agreement with this. To obtain an impartial calibration,

* *Contributions from the Mount Wilson Observatory, Carnegie Institution of Washington*, No. 636.

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the differences of the published M_s from the standard values S_1 for each spectral subclass should be multiplied by 2.22 for the main sequence and by 1.33 for giants. Values of S_1 (taking account of zero-point corrections) are given in Table 12.

The mean error θ'' of the corrected results is $\pm 0^m65$ for the main sequence and $\pm 0^m47$ for giants. This increase is much more than compensated by the removal of a serious systematic error. The probable error of a corrected spectroscopic parallax is ± 21 per cent and ± 16 per cent in the two cases.

The problem raised by this title is not easy, and its discussion has been complicated by the very different algebraic notation used by Dr. Strömberg and ourselves. Long conversations between the senior author and Dr. Strömberg have cleared up practically all the points of difference, but various matters deserve to be placed on record.

I. HOW NEARLY ARE THE CORRELATIONS LINEAR?

1. Since the calibration-curves used in finding spectroscopic absolute magnitudes are empirical, it cannot be hoped that they will succeed in giving a perfect linear correlation between the spectroscopic absolute magnitudes M_s and the true absolute magnitudes M. The actual question is: To what extent and over what range is the assumption of linear correlation a good approximation for the set of values of M_s given in *Mount Wilson Contribution* No. 511[2] so long as we confine ourselves to main-sequence stars of a narrow spectral range?

Both Dr. Strömberg and we have adopted Schlesinger's catalogue of trigonometric parallaxes[3] as a standard, disregarding its systematic errors, which are known to be very small. Our methods of analysis, though different in appearance because of differences in notation, are really very similar in principle and method. Strömberg uses the absolute magnitudes; we transform these into "reduced" parallaxes by the equation

$$s' = fs\,, \qquad t' = ft\,, \qquad f = 10^{0.2(m-S_0)}\,, \tag{1}$$

where s and t are the spectroscopic and trigonometric parallaxes, m the apparent magnitude, and S_0 a standard absolute magnitude for the spectral subclass, so that

$$M_s = S_0 + 5 + 5 \log s'\,, \qquad M = S_0 + 5 + 5 \log \pi'\,, \tag{2}$$

where $\pi' = f\pi$ is the reduced true parallax.

[2] Adams, Joy, Humason, and Brayton, *Ap. J.*, **81**, 187, 1935.

[3] *General Catalogue of Stellar Parallaxes* (2d ed.), Yale University Observatory, 1935.

If $\Delta M = a\Delta M_s$, $\pi' = \text{const } s'^a$. A linear relation in one case produces a curvature in the other; it will be seen (§ 4) that, for the range of practical importance, this curvature is insensible.

2. Two methods of analysis are possible: (*a*) to apply a correlation analysis to the deviations of the data for the individual stars from the mean for a spectral group or (*b*) to combine the stars into subgroups, thus eliminating the worst of the accidental errors, and study the means. Both methods may be applied to s' and t', as was done in our paper:[4] only the second is available for M_s and M, on account of the appearance of negative values of t.

The subgroups may be selected either by a "biased" criterion influenced by the errors of M_s (e.g., by M_s) or by an independent criterion uninfluenced by these (e.g., by $H = m + 5 \log \mu$). Strömberg[5] has used these criteria; we have employed s', and $\mu' = f\mu$ (A, § 14).

Since

$$H = S_0 + 5 \log \mu' , \qquad (3)$$

these groupings would be identical with Strömberg's for a single spectral subclass, where S_0 is fixed. In his spectral groups G0–G7 and G8–K2, S_0 ranges from 4.0 to 5.0 and from 5.1 to 5.9, while the range of H exceeds 15^m and 14^m. Selection by μ' is therefore practically, though not rigorously, equivalent to that by H. The dispersion of $M_s - S_0$ is small, and selection by M_s favors the earlier subclasses in the bright groups.

3. Seeking a purely objective method of division, we arranged our subgroups so that each contained one-fourth of the stars of each spectral subclass (to the nearest integer). To get more detail, we have now formed eight similar groups. For the separate subclasses the numbers are so small that the results are ragged, and we have taken weighted means for four spectral groups—so chosen that one coincides with Strömberg's G0–G7. The means for Strömberg's second group, G8–K2, and for all main-sequence stars together have also been taken, and also those for all giants together, including super- and subgiants.

[4] *Mt. W. Contr.*, No. 589; *Ap. J.*, **87**, 389, 1938 (referred to hereafter as "A").

[5] *Mt. W. Contr.*, No. 603; *Ap. J.*, **89**, 10, 1939 (referred to as "B"); *Mt. W. Contr.*, No. 628; *Ap. J.*, **92**, 235, 1940 (referred to as "C").

Table 1 gives the impartial grouping by μ', and Table 2 the "biased" grouping by s'. N is the number of stars; the means $\overline{s'}$, $\overline{t'}$, and $\overline{\mu'}$ are in units of 0\.001. Nine stars for which no proper motions were available are omitted in Table 1 and in forming $\overline{\mu'}$ in Table 2.

TABLE 1

GROUPING ACCORDING TO μ'

Spectrum and Number of Stars		I	II	III	IV	V	VI	VII	VIII
A1–F3 260	N $\overline{s'}$ $\overline{t'}$	29 91 62	32 92 70	32 97 90	35 103 86	35 103 112	33 116 106	32 114 120	32 128 146
F4–F9 276	N $\overline{s'}$ $\overline{t'}$	33 83 72	35 102 98	34 106 108	36 100 104	36 113 118	34 114 122	32 109 113	36 118 152
G0–G7 269	N $\overline{s'}$ $\overline{t'}$	29 101 87	32 100 90	32 104 104	38 110 113	38 110 117	32 111 128	33 109 131	35 133 172
G8–M5 330	N $\overline{s'}$ $\overline{t'}$	36 90 86	41 96 90	42 96 91	42 106 108	44 99 98	43 101 102	41 107 108	41 112 123
G8–K2 117	N $\overline{s'}$ $\overline{t'}$	12 86 84	14 98 104	15 94 82	15 108 117	16 94 101	15 97 96	14 106 92	16 107 126
Main Sequence 1135	N $\overline{s'}$ $\overline{t'}$	127 91.1 79.0	140 97.8 88.2	140 100.0 97.4	151 105.1 104.6	153 104.8 108.5	142 108.2 111.0	138 109.1 115.5	144 118.9 139.2
Giants 728	N $\overline{s'}$ $\overline{t'}$	93 58.9 52.3	89 82.2 72.4	92 92.5 82.0	88 99.0 105.9	91 98.2 110.4	91 106.2 114.2	90 109.2 110.9	94 141.6 145.0

The original calculations were carried to one more decimal place, which is omitted except for the general mean.

To extend the range, the outermost groups of stars were divided into as nearly equal parts as possible, with the results given in Table 3. Division *a* corresponds to the smaller value of μ' or s'. The numbers for G8–K2 are too small for profitable subdivision, nor was one made for the giants.

4. These data are plotted in Figures 1 and 2, the whole groups as dots, the half-groups as open circles. For the separate spectral

TABLE 2

GROUPING ACCORDING TO s'

Spectrum and Number of Stars		I	II	III	IV	V	VI	VII	VIII
A1–F3 260	N	29	32	32	35	35	33	32	32
	$\overline{s'}$	69	82	94	104	112	117	123	150
	$\overline{t'}$	50	80	76	104	101	115	119	142
	$\overline{\mu'}$	294	387	373	460	580	617	884	930
F4–F9 278	N	33	37	34	36	36	34	32	36
	$\overline{s'}$	69	91	97	104	109	114	121	138
	$\overline{t'}$	56	99	111	110	107	114	125	155
	$\overline{\mu'}$	415	802	1172	1084	781	790	1122	1648
G0–G7 270	N	30	32	32	38	37	33	32	36
	$\overline{s'}$	80	92	99	104	110	115	122	139
	$\overline{t'}$	65	80	96	119	114	132	143	147
	$\overline{\mu'}$	841	1050	1164	1664	1589	1707	1801	2475
G8–M5 332	N	36	41	42	44	43	43	41	42
	$\overline{s'}$	75	84	91	96	101	108	116	131
	$\overline{t'}$	71	87	91	92	104	102	122	128
	$\overline{\mu'}$	830	987	1061	1173	1231	1322	1900	1510
G8–K2 117	N	12	14	15	16	15	15	14	16
	$\overline{s'}$	72	85	92	96	100	106	111	128
	$\overline{t'}$	66	91	91	90	120	100	124	121
	$\overline{\mu'}$	595	1170	1017	1380	1790	1209	3158	1592
Main Sequence 1140	N	128	142	140	153	151	143	137	146
	$\overline{s'}$	73.5	86.7	94.5	100.6	106.5	112.4	119.0	137.1
	$\overline{t'}$	63.2	86.5	93.3	102.9	106.4	112.4	126.1	139.5
	$\overline{\mu'}$	633	848	983	1107	1095	1160	1556	1698
Giants 732	N	93	89	94	89	92	93	90	92
	$\overline{s'}$	50.0	79.8	86.5	96.5	102.2	107.3	117.4	170.5
	$\overline{t'}$	41.2	74.7	94.2	97.9	89.1	119.8	120.0	185.3
	$\overline{\mu'}$	167	506	583	541	554	795	886	1654

groups the plots show large irregularities, and that for the small group G8–K2 is too ragged to justify graphical treatment—or to give reliable results by any other method. For the whole main se-

quence the plots are much smoother and remarkably straight, especially for the impartial selection by μ'.

The straight lines have been derived by the analysis described in § 6 and are "impartial" lines for the grouping by μ' and regression lines for the grouping by s'. The deviations are considerable, but obviously mainly unsystematic. In a few cases (e.g., F4–F9, by μ')

TABLE 3

DIVISION OF THE OUTER GROUPS

Spectrum		Grouping by μ'				Grouping by s'			
		I*a*	I*b*	VIII*a*	VIII*b*	I*a*	I*b*	VIII*a*	VIII*b*
A1–F3	N	14	15	16	16	14	15	16	16
	$\overline{s'}$	96	84	138	115	66	72	143	156
	$\overline{t'}$	59	66	149	142	42	58	114	170
F4–F9	N	16	17	18	18	16	17	18	18
	$\overline{s'}$	74	89	115	123	62	80	127	148
	$\overline{t'}$	62	80	144	166	45	72	144	166
G0–G7	N	14	15	18	17	15	15	18	18
	$\overline{s'}$	98	104	123	143	75	87	130	149
	$\overline{t'}$	86	89	164	180	61	70	131	166
G8–M5	N	18	18	21	20	18	18	21	21
	$\overline{s'}$	93	88	110	114	72	78	126	136
	$\overline{t'}$	86	85	118	128	66	76	122	133
Main Sequence	N	62	65	73	71	63	65	73	73
	$\overline{s'}$	91	91	118	120	69	79	130	144
	$\overline{t'}$	76	82	136	143	56	70	127	152

a curve would fit better, but the deviations from a straight line are no worse than for the obviously random ones for A1–F3. We conclude, therefore, that the assumption of a linear correlation between $\overline{s'}$ and $\overline{t'}$ represents the data satisfactorily.

The dotted curves on the plots for "Main Sequence" are of the form $\overline{t'} = \text{const}\ \overline{s'}^{a}$ and correspond to a linear correlation between M and M_s. For the grouping by s' they are indistinguishable; for that by μ' the curve gives a slightly better fit. It would evidently require several thousand observations to permit a decisive choice.

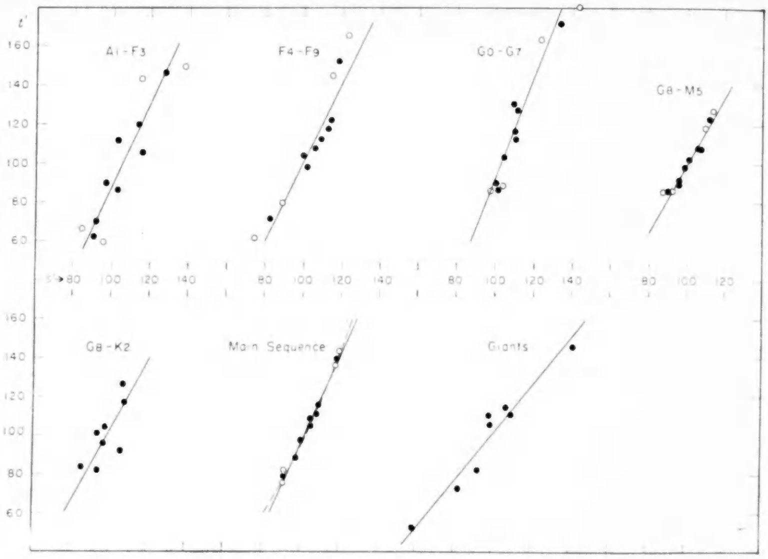

FIG. 1.—Grouping according to μ'. Abscissae are values of $\overline{s'}$; ordinates, values of $\overline{t'}$. Dots represent data from Table 1; circles, data from Table 3, division 1.

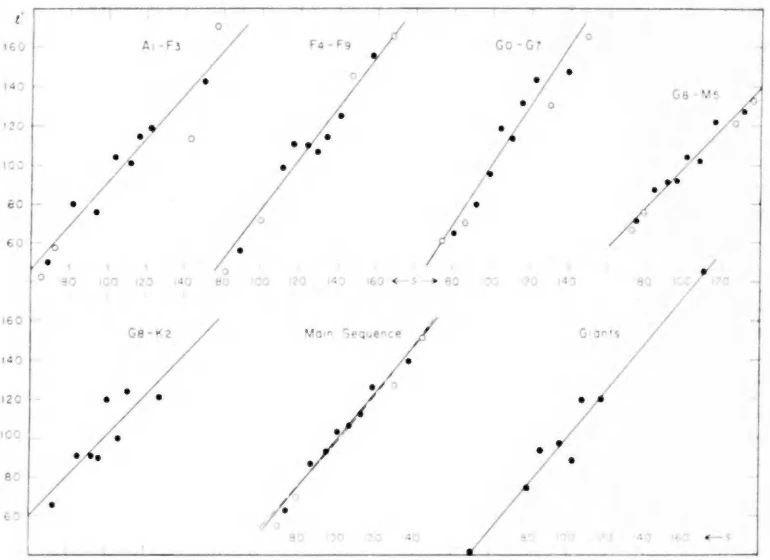

FIG. 2.—Grouping according to s'. Abscissae are values of $\overline{s'}$; ordinates, values of $\overline{t'}$. Dots represent data from Table 2; circles, data from Table 3, division 2.

For the stars which deviate even more from the mean than the outermost groups here considered—that is, by more than about 1^m in absolute magnitude—departures from the linear relations may be considerable. But these stars form a very small percentage of the whole; and it is probable that they are physically different—in composition or internal constitution—from normal main-sequence stars. In this case the ordinary spectroscopic determination of absolute magnitude may be expected to go astray (§ 16). If enough such stars were available, with good trigonometric parallaxes, a special calibration should be possible.

5. Of Strömberg's curves, two are nearly straight. For G8–K2, selected by M_s (C, Fig. 2), he informs us that the adopted curve results from the assumption that the three lower points represent a group of stars which are physically distinct from the rest. In this case an alternation of low slopes within the groups and steep slopes connecting them is to be expected.[6] This hypothesis is permissible, but in view of the small number of stars involved, not conclusive. A straight line represents all the points adequately.

In drawing the curve for G0–G7, grouped by H (C, Fig. 1), he has given great weight to the open circles representing a few stars with large and well-determined trigonometric parallaxes. The values of M for these are accurately known; but those of M_s are as much subject to error as any others and much less accurate than the group means.

The deviation found by Strömberg from the straight line which represents the general run of all the 263 G stars is conditioned almost entirely by the uppermost circle, representing Groombridge 1830. But this star, with $M = 6.7$, is $2^m\!.1$ fainter than the average for class G5 and is evidently a subdwarf. It is also peculiar in spectrum, for its tabular M_s is 6.3—fainter by $0^m\!.7$ than any other star of classes G4–G6 in *Mount Wilson Contribution* 511. The more pronounced subdwarf Cin 2019 ($15^h4^m\!.7$, $-15°54'$, $8^m\!.9$, G2, μ $3''\!.74$, t $0''\!.040 \pm 0.004$, M 6.9) has $M_s = 4.4$. The linear correlation fails for it, in the opposite direction. Until more data for subdwarfs are available, it is doubtful whether the spectroscopic calibration can be applied to them with any accuracy.

[6] Cf. Strömberg, B, Figs. 3 and 4.

The differences between our curves and Strömberg's therefore depend upon questions of judgment in weighting the data for marginal and peculiar cases, where agreement cannot be reached by irrefutable arguments. For the great mass of the data the assumption of linear correlation appears to be a fully adequate approximation.

II. REDETERMINATION OF THE DISPERSION CONSTANTS

6. The data of Tables 1 and 2 deserve more than the summary discussion which we gave in A. In each spectral group the eight values of $\overline{s'}$, etc., have been given equal weight;[7] and the residuals from their general means, $v = \overline{s'} - \overline{\overline{s'}}$, $v' = \overline{t'} - \overline{\overline{t'}}$, $v'' = \overline{\mu'} - \overline{\overline{\mu'}}$, have been subject to a linear correlation analysis.[8] The regression lines are $v' = a_1 v$ (grouping by $\overline{s'}$) and $v' = a_2 v$ (grouping by $\overline{t'}$), where

$$a_1 = \frac{\Sigma vv'}{\Sigma v^2}, \qquad a_2 = \frac{\Sigma v'^2}{\Sigma vv'}. \tag{4}$$

The correlation coefficient, $r = \sqrt{a_1 a_2}$, measures the approach to an actual linear relation (in the present case, the degree to which irregularities have been removed by averaging). The geometric mean of the regression lines is

$$v' = av, \qquad \text{where } a = \sqrt{a_1 a_2}. \tag{5}$$

The residuals from the regression lines give "probable errors" for a_1 and $1/a_2$ which may be taken as a lower limit of their real uncertainty. The percentage probable errors are always nearly equal and may thus be applied also to a.

The results of this analysis are given in Table 4. For comparison with other data the results are expressed as fractions of the mean values $\overline{\overline{s'}}$, $\overline{\overline{t'}}$, $\overline{\overline{\mu'}}$; the dispersions σ_s, σ_t, σ_μ of v, v', v'' are divided by them, while

$$\frac{\overline{\overline{t'}}}{\overline{\overline{s'}}\, a_{st}}$$

[7] The weights previously assigned were less for the groups of large $\overline{t'}$; but the weighting system depends on the reduced probable error r' and not on r'/p', as it should do (if possible); hence, it underweights these groups.

[8] Cf. Strömberg, C, "Correlation Theory."

is the ratio of the proportional increments

$$\frac{\Delta \overline{s'}}{\overline{\overline{s'}}} : \frac{\Delta \overline{t'}}{\overline{\overline{t'}}} .$$

This quantity and the corresponding one for $\overline{t'}$ and $\overline{\mu'}$ are derived from the geometric mean of the regression lines and may be taken

TABLE 4

ANALYSIS BY SUBGROUPS

Spectra	A1–F3	F4–F9	G0–G7	G8–M5	G8–K2	Main Sequence	Giants
			Grouping by μ'				
$\overline{\overline{s'}}$	105.5	105.6	109.7	100.9	98.9	104.4	98.4
$\overline{t'}$	99.2	110.9	117.6	100.9	100.4	105.4	99.1
$\sigma_s/\overline{\overline{s'}}$	0.115	0.099	0.087	0.065	0.073	0.074	0.224
$\sigma_t/\overline{t'}$	0.259	0.190	0.214	0.114	0.142	0.160	0.272
r_{st}	0.933	0.932	0.959	0.985	0.730	0.992	0.966
$\overline{t'}/a\overline{\overline{s'}}$	0.44	0.52	0.41	0.58	0.56	0.460	0.823
P.e.	±0.05	±0.05	±0.03	±0.04	±0.13	±0.016	±0.062
a_i/a	0.99	1.01	1.00	1.01	1.05	1.002	0.970
			Grouping by s'				
$\overline{\overline{s'}}$	106.4	105.3	107.6	100.2	98.7	103.8	101.3
$\overline{\overline{t'}}$	98.3	109.6	112.1	99.5	100.4	103.8	102.8
$\overline{\overline{\mu'}}$	558	977	1536	1314	1489	1135	711
$\sigma_s/\overline{\overline{s'}}$	0.222	0.185	0.159	0.168	0.158	0.178	0.320
$\sigma_t/\overline{\overline{t'}}$	0.274	0.235	0.246	0.175	0.186	0.214	0.381
$\sigma_\mu/\overline{\overline{\mu'}}$	0.396	0.351	0.314	0.250	0.477	0.288	0.574
r_{st}	0.970	0.962	0.951	0.973	0.828	0.991	0.983
$\overline{s'}a_1/\overline{\overline{t'}}$	1.20	1.23	1.49	1.02	1.01	1.187	1.167
P.e.	±0.08	±0.09	±0.13	±0.07	±0.17	±0.046	±0.060
$r_{t\mu}$	0.908	0.905	0.935	0.905	0.817	0.977	0.981
$\overline{t'}/a\overline{\overline{\mu'}}$	0.69	0.67	0.79	0.70	0.39	0.74	0.66
P.e.	±0.08	±0.08	±0.09	±0.09	±0.08	±0.05	±0.04

to represent the true impartial line. For $\overline{t'}$ and $\overline{s'}$, grouped by s', only the regression line is significant.

It should be noticed that the "impartial-relation" line defined by Strömberg and given by equation (5) and the "impartial" line adopted in A (p. 5) are not the same. The former applies to the case where nothing is known about the residuals v, v' for individual stars except their statistical distribution. We assumed that they might be analyzed into independent and uncorrelated components, w_1, x_1, y_1, satisfying the equations

$$v = l\overline{\overline{s'}}(w_1 + y_1)\,, \qquad v' = \overline{\overline{t'}}(w_1 + x_1)\,. \tag{6}$$

For the regression lines we then have

$$a_1 = \frac{\overline{\overline{t'}}}{\overline{\overline{s'}}}\,\frac{W_1^2}{l(W_1^2 + Y_1^2)}\,, \qquad a_2 = \frac{\overline{\overline{t'}}}{\overline{\overline{s'}}}\,\frac{W_1^2 + X_1^2}{lW_1^2}\,, \tag{7}$$

and the slope of the impartial-relation line is

$$a = \frac{\overline{\overline{t'}}}{l\overline{\overline{s'}}}\sqrt{\frac{W_1^2 + X_1^2}{W_1^2 + Y_1^2}}\,,$$

while the correlation coefficient is given by

$$r^2 = \frac{W_1^4}{(W_1^2 + X_1^2)(W_1^2 + Y_1^2)}\,. \tag{8}$$

This impartial line is that obtained by taking means for groups selected in such a way that the mean values of x_1 and y_1 approach zero, while that of w_1 does not.

$$\bar{v} = \overline{\overline{s'}}\,l\overline{w_1}\,, \qquad \overline{v'} = \overline{\overline{t'}}\,\overline{w_1}\,.$$

The slope of the corresponding line is

$$a_i = \frac{\overline{\overline{t'}}}{l\overline{\overline{s'}}} = a\sqrt{\frac{W_1^2 + Y_1^2}{W_1^2 + X_1^2}}\,. \tag{9}$$

When the weights are unequal, BX_0^2 must be substituted for X_1^2 and Y_0^2 for Y_1^2, or $BX_0^2/\bar{p}$ and $Y_0^2/\bar{p}$, if our analysis is based on means for groups of weight $\bar{p}$.

Taking these quantities from Table 5, we find the values of a_i/a given in Table 4. These corrections have been taken into account in Table 6.

The correlation between s' and t' is of little interest, as it depends on the trigonometric errors. That between s' and π' (or very nearly between M_s and M) may be found by setting $X_1 = 0$ in equation (8), giving

$$r_0^2 = \frac{W_1^2}{W_1^2 + Y_1^2} . \tag{10}$$

7. For comparison with these results we need those derived from the same data and the same spectral groups by our earlier method.

TABLE 5

ANALYSIS BY SQUARES AND PRODUCTS OF RESIDUALS

Sp.	N	$[p]$	$W_1^2+BX_0^2$	BX_0^2	W_1^2	$l_1W_1^2$	$l_1^2(W_1^2+Y_0^2)$	l_1	W_1	Y_0
A1–F3 . . .	260	188.5	0.282	0.160	0.122	0.063	0.053	0.51	0.35	0.28
F4–F9 . . .	278	216.4	.266	.114	.152	.050	.039	.33	.39	.45
G0–G7 . . .	270	221.3	.242	.100	.142	.051	.036	.36	.38	.37
G8–M5 . . .	332	427.2	.138	.050	.088	.033	.033	.37	.30	.39
G8–K2 . . .	117	124.6	.204	.076	.128	.033	.030	.26	.36	.56
Main Sequence .	1140	1053.4	.212	.093	.119	.046	.038	.38	.34	.38
Giants . . .	732	212.9	0.877	0.608	0.269	0.170	0.164	0.63	0.52	0.38

These can be very easily obtained by summing the proper columns of Table 3 (A, p. 21) and applying equations (24)–(26) of A (pp. 12–13) to the means. They are given here in Table 5 (in the earlier notation). The giant stars are added for comparison.

8. Consider first the dispersion σ. For a variable with a normal distribution and the standard deviation ϵ, the mean values for eight equal groups are $\pm 1.69\,\epsilon$, $\pm 0.90\,\epsilon$, $\pm 0.49\,\epsilon$, and $\pm 0.16\,\epsilon$, giving $\sigma = 0.974\,\epsilon$.

For the individual values of $s'/\overline{\overline{s'}}$, equation (26) of A gives $\epsilon^2 = l_1^2(W_1^2 + Y_0^2)$. The ratios of $\sigma_s/\overline{\overline{s'}}$ (Table 4) to 0.974ϵ are 0.99, 0.96, 0.86, 0.95, 0.94, and 0.94 for the six spectral groups. The difference from unity probably arises from departures from a normal distribution.

If the accidental errors average out, we should have $\sigma_t/\bar{\bar{t}}' = \sigma_\pi/\bar{\bar{\pi}}' = 0.974W_1$ or 0.335 for "Main Sequence." The actual value is 0.62 times this—showing how far selection by s' is from picking out the stars in order of their real parallaxes. For "Main Sequence" grouped by μ', the values of $\sigma_s/\bar{\bar{s}}'$ and $\sigma_t/\bar{\bar{t}}'$ are 0.41 and 0.51 times those just calculated. The diminution for s' is partly caused by the elimination of accidental errors, which enter to their full amount in ϵ^2; but that for t' shows that selection by μ' (or H) is a weak—though impartial—method for separating stars of different absolute magnitude.

TABLE 6

COMPARISON OF CALCULATED SLOPES

Sp.	Regression Line		Impartial Line		Impartial (Corrected)	
	$W_1^2/l_1(W_1^2+V_0^2)$	$\bar{\bar{s}}'a_1/\bar{\bar{t}}'$	$1/l_1$	$\bar{\bar{s}}''a_1/\bar{\bar{t}}'$	$1+V_0^2/W_1^2$	a_1/a
A1–F3....	1.18±0.12	1.20±0.08	1.95±0.30	2.23±0.23	1.65	1.88
F4–F9....	1.28± .14	1.23± .09	3.01± .39	1.95± .20	2.35	1.57
G0–G7....	1.40± .13	1.49± .13	2.78± .32	2.46± .20	1.98	1.65
G8–M5....	1.00± .10	1.02± .07	2.70± .29	1.76± .09	2.70	1.71
G8–K2....	1.12± .22	1.01± .17	3.86± .83	1.87± .44	3.44	1.76
Main Sequence..	1.19± .06	1.19± .05	2.61± .16	2.17± .08	2.20	1.83
Giants....	1.03±0.05	1.17±0.06	1.59±0.15	1.25±0.09	1.54	1.07

The most important comparison is that of the slopes of the regression and impartial lines. These are given in Table 6 with their *mean* errors. Those of $1/l_1$ and $W_1^2/l_1(W_1^2 + V_0^2)$ represent the errors to be expected from random sampling alone (A, § 7) and are probably too small. The last two columns give the slope which the impartial line would have if the spectroscopic calibration, still made by means of the regression lines, were altered so that the slope of the new regression lines was unity.

9. The results for the regression lines agree so well as to show that only a small part of the errors affects the two methods of calculation differently. They indicate that a considerable improvement of the calibration of spectroscopic parallaxes, by the existing method, is possible—probably on account of the much greater

volume of data now available. The differences in slope for the spectral groups must be mainly real. For the giants the influence of the trigonometric errors is proportionately much greater, and so is the discordance.

For the impartial line the slopes for the various groups differ widely. With the new calibration the agreement would be much improved, especially for the second method.

There is, however, a systematic difference between the results of the two methods, which is much too great to be attributed to accidental error. Two explanations, at least, are possible.

a) This discrepancy can be removed by assuming that the trigonometric probable errors given in Schlesinger's catalogue are too small. Introducing into Table 5 the value of l found in Table 4, 0.460 ± 0.016 (m.e.) for the main-sequence stars together, we have, restoring decimals rounded off, $l_1 W_1^2 = 0.0455 \pm 0.0032$ (mean error of sampling). Hence, $W_1^2 = 0.099 \pm 0.008$. But $W_1^2 + BX_0^2 = 0.212 \pm 0.009$, whence $BX_0^2 = 0.113 \pm 0.012$ (assuming that the statistical errors are propagated as though they were independent). This is 1.22 ± 0.13 (m.e.) times the value in Table 5 and would be obtained by increasing Schlesinger's probable errors by 10.5 ± 6.5 per cent.

For the giants we find similarly $BX_0^2 = 0.677 \pm 0.055$, and an increase of the probable errors of 5.1 ± 4.3 per cent. So large an increase is hardly admissible, in view of the great skill, care, and caution employed in their determination, and the statistical uncertainties of the proposed correction are serious.

b) It is possible that μ' (or H) is correlated in some way with the spectroscopic criteria for absolute magnitude, so that selection by H is not statistically equivalent to that by M. A suggestion by Dr. Strömberg has led us to re-examine this possibility (at first dismissed as very improbable); but, before it can be adequately discussed, certain other matters must be considered.

III. RELATION BETWEEN REDUCED PARALLAX AND PROPER MOTION

10. Table 4 shows a strong correlation between $\overline{\mu'}$ and $\overline{l'}$ (grouped by s'). This is not the familiar relation between H and M for the

stars at large, regardless of spectral type, but one between tangential velocity and difference of absolute magnitude from the mean for the main sequence (or for the giants) of a given spectral subclass. A similar relation has been found by Strömberg (B, pp. 9–13), who has shown that the regression line of $\overline{M}$ as a function of $\overline{H}$, for stars of similar spectrum, can be represented by two linear portions for giants and dwarfs. For the latter, his data (Figs. 1 and 2) indicate a slope $\Delta M/\Delta H$ of +0.38 for G0–G7 and of +0.22 for G8–K2. Our slopes 0.79 and 0.39 refer to the impartial line and are steeper. Strömberg's means for giants and dwarfs (B, Table 1, omitting the

TABLE 7

IMPARTIAL RELATION OF M AND H

Sp.	A1–F3	F4–F9	G0–G7	G8–K5	K6–M5
$\overline{S}_0$	2.3	3.4	4.4	6.0	8.7
$\overline{\overline{t'}}$	98	110	112	98	102
$\overline{\overline{\mu'}}$	558	977	1536	1314	1205
(M)	2.3	3.6	4.7	5.9	8.8
(H)	1.0	3.3	5.3	6.6	9.1

intermediate mixed group) give, for G0–G7, $\Delta M = +4.6$, $\Delta H = +7.1$, and $\Delta M/\Delta H = 0.65$, and for G8–M2, +4.9, +6.7, and 0.73. These values are probably close to the slope of the impartial lines. Further information may be obtained from the values of $\overline{\overline{t'}}$ and $\overline{\overline{\mu'}}$. Dividing the large group G8–M5, we find the results shown in Table 7. The values (M) and (H) are computed from S_0, $\overline{\overline{t}}$, and $\overline{\overline{\mu}}$ without the corrections for the effect of dispersion within the group (for which no data were easily available). They should, however, be adequate to determine $\Delta M/\Delta H$, which is found to be 0.80. The mean value from comparison of giants and dwarfs is 0.69; the means found in Table 4 are 0.74 for the main sequence and 0.66 for the giants. For an impartial selection, the change in M with H (and hence in log T with M) is very nearly the same, whether we compare stars in different parts of the main sequence, giants and dwarfs of the same spectral type, or main-sequence stars or giants of the same subclass but differing from the average brightness. No evidence has previously been available in the third case.

IV. POSSIBLE PHYSICAL CAUSES OF THE "ERRORS"

11. The magnitude of the "accidental" error of impartially calibrated spectroscopic magnitudes is somewhat disquieting.

It does not seem probable that errors of observation can be so great. But the spectroscopic values are necessarily calibrated to fit the general run of the stars. Large discordances may arise from deviations of the physical conditions in individual stars from the average, and these would persist even though the calibration was based upon any number of stars with precisely known parallaxes and though all line-pairs gave consistent results.

It is well known that the spectroscopic determination of absolute magnitude depends on the mass-luminosity relation. Let L be the bolometric luminosity of any star, M its mass, r its radius, J its surface brightness, and g the surface gravity—all in "solar units." Then

$$L = Jr^2 , \qquad g = \frac{\mathrm{M}}{r^2} , \qquad L = J\frac{\mathrm{M}}{g} . \tag{11}$$

The structure of a stellar atmosphere should be uniquely determined if J, g, and the atomic composition are given. Hence, the spectral class, as defined by any definite and consistent criteria, should be represented by a function

$$Sp = f_1(J, g, C') , \tag{12}$$

where C' is a shorthand expression for the effects of composition (which may involve many parameters).

The spectral characteristics from which absolute magnitudes are derived (and which we may call Ab) should be expressible by a different function

$$Ab = f_2(J, g, C') . \tag{13}$$

For atmospheres of fixed composition and specified Sp and Ab, equations (12) and (13) suffice to determine the values of J and g (with the abstract possibility of multiple solutions—which do not seem to occur in practice); but substitution of these in equation (11) determines only the ratio L/M.

To go farther, we must consider the internal constitution of the

star. If the composition of this portion were fixed, and also the equation of state and the laws governing opacity and liberation of subatomic energy, Vogt's Theorem would be applicable. Such stars form a one-parameter family, and their absolute magnitudes, masses, etc., should depend on the spectral class alone (as appears to be the case in the Praesepe cluster). The differences among the stars of a given spectral subclass must therefore depend on differences of composition.

12. The mass-luminosity relation, which may be derived from a study of radiative equilibrium, without inquiry regarding the source of stellar energy, may be expressed with good approximation by the equation

$$L = C\mathrm{M}^{a}J^{b}. \tag{14}$$

Eddington's theory (based on Kramers' law of absorption) gives $b = \frac{1}{5}$, which we will adopt.[9] Our work on stellar masses[10] shows that $a = 3.33$ represents adequately all the existing data, including both main-sequence stars and giants. The coefficient C is again a "portmanteau" expression for the effects of internal distribution of density and of composition. It changes greatly with the abundance of hydrogen.

Since a has been determined empirically from observations in which main-sequence stars have overwhelming weight, the adopted value takes into account the average of any changes which may exist in the density distribution and the hydrogen content along this sequence—leaving C to represent the effect of differences from this mean.

Eliminating M between (11) and (14), we have

$$\left.\begin{aligned} L &= C^{1/(1-a)}g^{a/(1-a)}J^{(b-a)/(1-a)} = C^{-0.43}g^{-1.43}J^{1.34}\,, \\ \mathrm{M} &= C^{1/(1-a)}g^{1/(1-a)}J^{(b-1)/(1-a)} = C^{-0.43}g^{-0.43}J^{0.34}\,, \\ r^2 &= C^{1/(1-a)}g^{a/(1-a)}J^{(b-1)/(1-a)} = C^{-0.43}g^{-1.43}J^{0.34}\,. \end{aligned}\right\} \tag{15}$$

These equations follow directly from the form (14) for the mass-luminosity relation. When J is fixed, a star which is brighter than

[9] He gives $T_e^{4/5}$, but $J \propto T_e^4$.

[10] Russell and Moore, *The Masses of the Stars*, University of Chicago Press, 1940. (The value 3.33 follows easily from that of eq. [94], p. 89.)

normal stars of the same mass is fainter than normal stars of the same surface gravity.

13. We must now consider the effects of variations of g, J, and C' upon the spectral characteristics *Sp* and *Ab*. Milne's remarkably good approximate theory[11] of stellar atmospheres treats them as isothermal layers and derives the numbers N_0 and N_1 of neutral and ionized atoms above the level where the electron pressure is P, and also the optical depth τ of this level.

For the present purpose we may discuss the simplified case where only one component of the atmosphere (or a group of elements of similar ionization potential) is undergoing ionization; but we have to consider two sources of opacity: (*a*) the interaction of atoms of the "active" element with the electrons produced from it, and (*b*) that of neutral hydrogen atoms with these electrons. Wildt has shown[12] that the photoelectric ionization of negative hydrogen ions produces a large part of the total opacity in the cooler main-sequence stars.

An approximate theory of both processes has been given by one of us.[13] The resulting equations (*op. cit.*, p. 248) may be written

$$\mu g N_1 = 2P - (1 - \epsilon) K \ln\left(1 + \frac{P}{K}\right),$$

$$\mu g N_0 = \frac{P^2}{K} - (1 - \epsilon)\left\{P - K \ln\left(1 + \frac{P}{K}\right)\right\},$$

$$\tau_1 = f_1(\lambda, T) K N_0,$$

$$\mu g \tau_2 = f_2(\lambda, T)\left\{\frac{4}{3}\frac{P^3}{K} + (1 + \epsilon) P^2\right\}.$$

Here ϵ is the fraction (by *number*) of all the atoms which are active, μ is the mean atomic weight, excluding the free electrons, P the electron pressure, and K the dissociation constant for ionization of the active element. The forms of $f_1(\lambda, T)$ and $f_2(\lambda, T)$ do not concern us here. There is good evidence that most of the atoms are of hydrogen, so that ϵ is small and μ approximately 1.

[11] *M.N.*, **89**, 17 and 157, 1928.

[12] *Ap. J.*, **89**, 295, and **90**, 611, 1939.

[13] H. N. Russell, *Mt. W. Contr.*, No. 477; *Ap. J.*, **78**, 239, 1933.

14. *Case 1.*

If J is constant (and hence also T and K) and if the opacity is due to the active element alone, the intensity of its lines is practically unaffected by its abundance ϵ. Changes in gravity do not affect N_0 or the arc lines; N_1 varies as $g^{-1/2}$, and the enhanced lines are stronger in giants.[14]

Of two stars, with the same J and C', star 2, with the smaller g, will be classified as of earlier spectral type than star 1; but star 3 with this same g and a slightly smaller J will be of the same spectral type as star 1, judged by the standard criteria Sp.

This change will make the ionization of those elements whose lines are influential in the ordinary definition of spectral class the same in star 1 as in star 3. For elements of easier or more difficult ionization (which we may suppose to be present in small quantities without vitiating our analysis) the ionization will differ in the two stars. The lower pressure in star 3 will increase the ionization for all elements alike; the lower temperature will decrease it to a degree increasing with the ionization potential. The familiar absolute magnitude effects are thus explained. Several investigators have found a good agreement between theory and fact.

15. *Case 2.*

If the opacity arises wholly from $\overline{H}$ ions, we must assume $\tau_2 =$ constant, at the photosphere. With fixed λ and T,

$$\frac{d}{dP}(gN_1) = KQ\,, \qquad \frac{d}{dP}(gN_0) = PQ\,,$$

$$\frac{d}{dP}(g\epsilon\tau_2) = 2P(P+K)Qf(\lambda, T)\,,$$

where

$$Q = \frac{2P + (1+\epsilon)K}{K(P+K)}\,.$$

We then find easily, for various degrees of ionization of the active element, the results given in Table 8. When the active element is highly ionized, the dependence of N_1 and N_0 on g is the same as in

[14] *Mt. W. Contr.*, No. 477, pp. 18–20; *Ap. J.*, **78**, 256–258, 1933.

Case 1. Otherwise, the absolute-magnitude effect persists, but in somewhat diminished amount. Its amount varies as $(\epsilon g)^n$, where n ranges from -0.33 to -0.50 as K goes from 0 to ∞.

As in Case 1, if we compare two stars of the same J but with different values of g, there will be a difference of spectral class, which may be removed by diminishing J for the star with smaller g. The absolute-magnitude effects will appear as before, and their amount will evidently be approximately proportional to the change in N_1/N_0 between star 1 and star 2; that is, to $g^{-1/2}$ when the hydrogen produces none of the opacity, and to $(\epsilon g)^n$ when it produces all of it.

TABLE 8

Ionization	Incipient	Halfway	Complete
K	0	P	∞
$\dfrac{d \log (gN_1)}{d \log (g\epsilon\tau_2)}$	$+\frac{1}{3}$	$+0.45$	$+\frac{1}{2}$
$\dfrac{d \log (gN_0)}{d \log (g\epsilon\tau_2)}$	$+\frac{2}{3}$	$+0.84$	$+1$
N_1 varies as	$\epsilon^{1/3}g^{-2/3}$	$\epsilon^{0.45}g^{-0.55}$	$\epsilon^{1/2}g^{-1/2}$
N_0 varies as	$\epsilon^{2/3}g^{-1/3}$	$\epsilon^{0.84}g^{-0.16}$	ϵ

16. There is no invariant relation between changes in ϵ and in the constant C in the mass-luminosity relation (14), for the former depends on the abundance of the metals relative to hydrogen in the atmosphere, and the latter mainly upon the similar abundance ϵ_0 in the interior. It is plausible to assume that ϵ and ϵ_0 will vary in the same sense if not to the same degree; but certain facts, especially the presence of *Li* and *Be* in the solar atmosphere, suggest that there is actually little or no mixing of the surface and the interior.

The theoretical dependence of C upon the abundance of hydrogen and helium has been discussed by Bengt Strömgren.[15] If no helium is present, he finds

$$L = \text{const } (\mu\beta)^{7.5}\bar{l}(1 - X_0^2)^{-1}M^{5.5}r^{-0.5} ,$$

[15] *Ap. J.*, **87**, 520, 1938.

where l is the guillotine factor, X_0 the abundance of hydrogen by weight, $\mu = 2/(1 + 3X_0)$, and β is Eddington's factor. With the aid of (11), this becomes

$$L = \text{const } (\mu\beta)^6 l^{0.8}(1 - X_0^2)^{-0.8} M^{4.4} J^{0.2} . \tag{16}$$

If μ' is the average atomic weight of a heavy atom,

$$\epsilon_0 = \frac{1 - X_0}{1 + (\mu' - 1)X_0} . \tag{17}$$

We may adopt $\mu' = 20$. With M and J constant and $\beta = 1$ (a close approximation for main-sequence stars), we find

$$-\frac{d \log \epsilon_0}{d \log X_0} = \frac{20X_0}{(1 - X_0)(1 + 19X_0)} ,$$

$$-\frac{d \log L}{d \log X_0} = \frac{1.6X_0^2}{1 - X_0^2} + \frac{18X_0}{1 + 3X_0} + 0.8 \frac{d \log l}{d \log X_0} .$$

The ratio of the functions of X_0 has the values 1.80, 2.08, 2.10, 1.92, and 1.60, when X_0 is 0.1, 0.2, 0.3, 0.4, and 0.5. The values of X_0 found by Strömgren lie between 0.11 and 0.45.

We have then, approximately,

$$\frac{d \log L}{d \log \epsilon_0} = 2 + 0.8 \frac{d \log l}{d \log \epsilon_0} .$$

To calculate the last term theoretically would be laborious, but Strömgren finds for ζ Herculis A (for which $J = 1$, $\log M = -0.02$ in solar units) $\log l = 0.41$ and $X_0 = 0.11$ (whence $\epsilon_0 = 0.29$), while for the sun, $\log l = 0.75$, $X_0 = 0.36$, and $\epsilon_0 = 0.082$. Hence $\Delta \log l = -0.34$, when $\Delta \log \epsilon_0 = +0.55$. We may then adopt, as a working approximation,

$$C = C_0 \epsilon_0^{1.5} .$$

The changes of μ and l with X_0 along the main sequence are taken into account by the adoption of the empirical exponent 3.3 for M. We then have, by (15),

$$L = C_0 \epsilon_0^{-0.65} g^{-1.43} J^{1.34} , \tag{18}$$

so that for stars containing little or no helium, a change in ϵ_0 should produce about 45 per cent as great a change in L as an equal percentage change in g.

If the atmospheric opacity arises wholly from negative hydrogen ions, equal percentage changes in ϵ and in g produce the same absolute-magnitude effect in the spectra. But a change in ϵ_0 produces a much smaller change in the absolute magnitude than one in g. Hence, if ϵ is proportional to ϵ_0, the spectra will be much more sensitive to change in absolute magnitude arising from differences in composition than from differences in gravity.

If hydrogen adds nothing to the atmospheric opacity, only changes in g produce absolute-magnitude effects in the spectra. Since the contribution of hydrogen ions varies with the conditions in the atmosphere, no exact proportionality between differences in absolute magnitude and the spectroscopic effects can be anticipated in general. If the two types of opacity are in a fixed ratio for a given spectral class, there should be a definite correlation between absolute magnitude and spectrum, within the group, so long as the abundance of the metals in the interior is closely correlated with that in the atmosphere.

If, however, the abundance of the metals in the atmosphere is not correlated with that in the interior, considerable differences in the absolute-magnitude criteria may appear in stars of the same absolute magnitude and create discordances in the spectroscopic parallaxes, which arise from causes inherent in the stars and cannot be removed by any refinement of observation. The presence of helium may introduce further irregularities. It is possible that a considerable part of the "accidental error" V arises in this way. The question cannot be settled until more accurate methods for the quantitative analysis of stellar atmospheres (especially for hydrogen) have been developed and applied to many stars.

V. SIGNIFICANCE OF GROUPING BY H

17. Study of the discordances mentioned in § 9 may begin with examining the assumptions regarding correlation upon which the equations at the end of § 6 are based.

If π' is the "reduced" value of the true parallax, these may be written

$$t' = \pi' + \overline{\overline{t'}} x_1 , \qquad \pi' = \overline{\overline{t'}}(1 + w_1) .$$

The assumption that the trigonometric errors are accidental leads to $\overline{\overline{x_1}} = 0$; the adoption of the trigonometric system as a standard gives $\overline{\overline{w_1}} = 0$. Let us now define y_1 by the equation

$$s' = \overline{\overline{s'}}\{1 + l(w_1 + y_1)\}$$

and assume that the regression line of s' upon π' is straight. Then, for any narrow range in π', we will have

$$\overline{s'} - \overline{\overline{s'}} = \frac{\overline{\pi'} - \overline{\overline{\pi'}}}{A} = \frac{\overline{\overline{t'}}\,\overline{w_1}}{A} .$$

But

$$\overline{s'} = \overline{\overline{s'}}\{1 + l(\overline{w_1} + \overline{y_1})\} .$$

Hence $l = \overline{\overline{t'}}/A\overline{\overline{s'}}$; and $\overline{y_1} = 0$ for all values of w_1, so that $\overline{\overline{w_1 y_1}} = 0$. Absence of correlation between w_1 and y_1 thus follows from the assumption of a straight regression line, without knowing anything more about the origin of y_1.

The conclusion that $\overline{\overline{x_1 y_1}} = 0$, $\overline{\overline{x_1 w_1}} = 0$ depends on the assumption that the results of accidental errors of measurement on parallax plates will not be correlated with properties of the stars observed or of their spectra, which seems thoroughly safe. But if $\overline{\overline{x_1 y_1}} = \overline{\overline{x_1 w_1}} = \overline{\overline{w_1 y_1}} = 0$, our analysis in A is valid.

We have no evidence as to the regression line of s' on π', but we do know that the "impartial" line for groupings by μ' is sensibly straight. If groupings by μ' and by π' (that is, by H and M) are statistically equivalent, the results of Table 5 may be accepted.

18. The equivalence of the two groupings, however, demands careful study. We have

$$H = M + 5 \log V_t + \text{const} , \tag{19}$$

where V_t is the star's tangential velocity. If this were constant, groupings by H and M would be identical. Actually, log V_t and M are correlated. The data of § 10 show that, approximately, for impartial selection

$$\Delta \log V_t = 0.07 \Delta M \,. \tag{20}$$

But there is no intelligible physical reason why the velocity of a star should be correlated directly with its luminosity, while a correlation between velocity and mass is to be expected, and one between velocity and composition is possible (e.g., if the high-velocity stars originated in a different region of space from the rest).

Equation (14) may be written

$$\log \mathrm{M} = -0.12M - 0.3 \log C - 0.06 \log J + \text{const} \,. \tag{21}$$

We may suppose that, statistically,

$$\log V_t = k \log \mathrm{M} + \log V_t' \,, \tag{22}$$

where V_t' is not correlated with M.

Equations (20), (21), and (22) will be consistent if $k = -0.58$, a rough determination. The plausible (but quite unproved) assumption of equipartition of energy gives $k = -0.5$. Adopting -0.58, we have roughly

$$H = 1.4M + 0.9 \log C + 0.2 \log J + 5 \log V_t' + \text{const} \,. \tag{23}$$

For a given spectral subclass, J is substantially constant. The dispersion in 5 log V_t' includes all the geometrical effects and is large.

It follows that selection by H is not equivalent to that by M. For large values of H, large values of C will be given preference—that is, stars that are brighter than normal for their mass.

If the composition of the atmosphere is correlated with that of the interior, the mean spectroscopic absolute magnitude S will also be affected—whether more or less than M will depend on the relative sensitiveness of the spectroscopic criteria to changes in g and C' (§ 16).

Hence the slopes of the lines obtained by selection according to H and to M may not be identical, but in the present state of the theory one cannot say which should be the greater.

19. Suppose that selection by any parameter P separates groups with different mean values of $\overline{s'}$ and $\overline{t'}$. Setting $u = v/\overline{\overline{s'}}$, and $u' = v'/\overline{\overline{t'}}$ in (6) we have,

$$\left.\begin{aligned} u' &= w_1 + x_1 , \qquad u = l_1(w_1 + y_1) , \\ \overline{\overline{u'^2}} &= W_1^2 + X_1^2 , \qquad \overline{\overline{uu'}} = l_1 W_1^2 , \qquad \overline{\overline{u^2}} = l_1^2(W_1^2 + Y_0^2) . \end{aligned}\right\} \quad (24)$$

Suppose, also, that selection by P gives a rectilinear plot of $\overline{s'}$ against $\overline{t'}$, with the equation $\bar{u} = l_2\overline{u'}$ and, to get a case capable of analytical discussion, that the correlation distribution of u and u' is normal.

Strömberg has shown[16] that this will be the case if the distributions for the individual variables are normal. According to his equation (3) and those just following, the exponential factor in the distribution function may be written

$$\exp - (Au^2 + 2Huu' + Bu'^2) ,$$

where

$$\left.\begin{aligned} A = C\overline{\overline{u'^2}} , \qquad B = C\overline{\overline{u^2}} , \qquad H = -C\overline{\overline{uu'}} , \\ 2C\{\overline{\overline{u^2}} \cdot \overline{\overline{u'^2}} - (\overline{\overline{uu'}})^2\} = 1 . \end{aligned}\right\} \quad (25)$$

If

$$q = u - l_2 u' , \qquad (26)$$

the axis of the ellipse $Au^2 + 2Huu' + Bu'^2 =$ constant, conjugate to $q = 0$, will be $q' = 0$, where

$$Cq' = (Al_2 + H)u + (Hl_2 + B)u' . \qquad (27)$$

The quantities q and q' are not correlated. Substituting from (24) and (25),

$$q' = \{(l_2 - l_1)W_1^2 + l_2 X_1^2\}u + l_1\{(l_1 - l_2)W_1^2 + l_1 Y_0^2\}u' . \qquad (28)$$

For groups selected by P, $\bar{q} = 0$, while $\overline{q'}$ is not zero. Moreover, q' is (except for a constant factor) the only linear function of u and u'

[16] *Pub. A.S.P.*, **51**, 231, 1939.

which will give the regression line $q = 0$. Hence, selection by P and by q' must be statistically equivalent.

20. Applying this to selection by μ', taking l_2 from Table 4, and $l_1 W_1^2$ and $l_1^2 Y_0^2$ from Table 5 and setting $X_1 = 0$ (since the trigonometric errors have been eliminated), we find by (28) for the main-sequence stars together,

$$\left.\begin{aligned} q' &= 0.0174(u' + 0.53u)\,, \\ &\text{and for the giants,} \\ q' &= 0.0247(u' + 2.12u)\,. \end{aligned}\right\} \tag{29}$$

Now by (23)

$$H = 1.4(M + 0.65 \log C + \ldots.)\,, \tag{30}$$

and by (15)

$$M = \text{const} + 1.08 \log C + 3.58 \log g - 3.36 \log J\,.$$

The ordinary calibration of spectroscopic parallaxes takes account of the variations of g and J. We found in § 16 that, according to circumstances, the spectroscopic criteria may be more than twice as sensitive to changes in C as in g, or hardly sensitive at all. Hence, we may write $M_s = M + a \log C$, where a may range from $+1$ to -1. Substituting in equation (30),

$$H = 1.4\left\{\left(1 - \frac{0.65}{a}\right) M + \frac{0.65}{a} M_s\right\}\,.$$

But $M = \text{const} + 2.17u'$, $M_s = \text{const} + 2.17u$. Hence, selection by H is equivalent to that by q' if $a = 1.88$ for the main-sequence stars and 0.96 for the giants. The first of these values is inadmissible theoretically; the second, at the limit of possibility. It appears, therefore, that the observed discrepancies between l_1 and l_2 (especially the better-determined value for the main sequence) cannot be accounted for by physical differences in selection by M and by H, at least on the basis of the theory here developed.

VI. ADOPTED VALUES OF THE CONSTANTS

21. No definite reasons have developed for giving the results of the analysis either by means for groups (Table 4) or by correlation methods (Table 5) any decisive preference over the other. We have therefore adopted the means of the two values of the slopes given in Table 6, despite the discordance. There is no obvious reason to prefer the slope of s' with respect to t' to its reciprocal; we have therefore taken means of the logarithms of the tabular values, with weights inversely as the squares of their own mean errors. The results are given in Table 9. The values of the recalibrated slope of the

TABLE 9

ADOPTED RESULTS: PARALLAXES

Sp.	$\frac{W_1^2}{l_1(W_1^2+Y_0^2)}$	$\frac{1}{l_1}$	$1+\frac{Y_0^2}{W_1^2}$	W_1 M.e.	Y_0
A1–F3.......	1.19	2.14	1.79	0.366 ± 0.014	0.324
F4–F9.......	1.25	2.32	1.86	.348 ± .016	.322
G0–G7.......	1.45	2.56	1.77	.369 ± .014	.324
G8–M5.......	1.01	1.90	1.88	.262 ± .015	.246
G8–K2.......	1.05	2.78	2.64	.309 ± .021	.396
Main Sequence	1.19	2.28	1.92	.327 ± .007	.314
Giants.......	1.10	1.37	1.24	0.495 ± 0.018	0.242

impartial line (fourth column) are in remarkable agreement for the main sequence, except for the weak group G8–K2.

We may now obtain improved values of the dispersion constants W_1 and Y_0. Adopting the constants of Table 9, the determinations of W_1^2, $l_1W_1^2$, and $l_1^2(W_1^2 + Y_0^2)$ given in Table 5 become multiples of W_1^2. Weighting them according to their statistical mean errors (which have already been found), we find the values of W_1 and Y_0 given in Table 9. The mean errors of W_1 are derived from the accordance of the three values of W_1^2, except for A1–F3 and G0–G7 and the giants, where the error to be expected from sampling is greater, and therefore is given. It is probable that the actual errors of determination are considerably greater.

The decrease of W_1 and Y_0 for G8–M5 is undoubtedly real. The group G8–K2 gives, as always, ragged results; it is really too small for a statistical study.

22. To pass from W_1 and V_0 to the corresponding dispersions in absolute magnitude they should, in the first approximation, be multiplied by 5 Mod = 2.171. A closer study demands some assumption about the distribution-functions of M and M_s. Strömberg assumes these to be normal, and we will follow him—both for comparison and because the higher-order corrections thus obtained are probably better than none. That is, we will determine the constants of that *normal* distribution of absolute magnitude which would lead to the constants derived for the parallaxes in Table 9.

Set

$$M = \bar{M} + w \,. \tag{31}$$

Then, by (2),

$$10\pi' = \exp a\,(M - S_0) = Ce^{aw}\,,$$

where

$$C = 10^{0.2(\bar{M}-S_0)}\,, \qquad a = \frac{1}{5\ \mathrm{Mod}} = 0.4605\,.$$

If w has a normal distribution, with mean error W, the mean value of e^{aw} is $e^{1/2a^2W^2}$, and that of $(e^{aw} - \overline{e^{aw}})^2$ is $e^{2a^2W^2} - e^{a^2W^2}$. Hence, if $\pi' = \overline{\pi'}(1 + w_1)$, we have

$$\overline{w_1^2} = W_1^2 = e^{a^2W^2} - 1$$

and also

$$10\overline{\pi'} = C(1 + W_1^2)^{1/2}\,.$$

$$\bar{M} = S_0 + 5 + 5 \log \overline{\pi'} - 2.5 \log (1 + W_1^2)\,.$$

These equations are exact if the postulates are fulfilled.

If, similarly, we assume normal distribution for u, and that

$$M_s - \bar{M}_s = u\,, \qquad s' = \overline{s'}(1 + u_1)\,, \tag{32}$$

we find

$$\overline{u_1^2} = \exp a^2U^2 - 1\,,$$

and also

$$\overline{w_1u_1} = \exp \tfrac{1}{2}a^2\{\overline{(w + u)^2} - W^2 - U^2\} - 1\,.$$

If we assume that

$$u = l(w + y) ,$$

the distribution of y being normal and not correlated with w, the distribution of u will still be normal, and we shall have

$$\overline{u_1^2} = \exp a^2l^2(W^2 + Y^2) - 1 ,$$

$$\overline{u_1w_1} = \exp a^2lW^2 - 1 .$$

In our previous analysis, we set $\overline{w_1^2} = W_1^2$, $\overline{u_1w_1} = l_1W_1^2$, $\overline{u_1^2} = l_1^2(W_1^2 + Y_0^2)$. Hence, we have

$$\left.\begin{aligned} a^2W^2 = ln(1 + W_1^2) &= W_1^2(1 - \tfrac{1}{2}W_1^2 + \ldots .) \\ a^2lW^2 \qquad &= l_1W_1^2(1 - \tfrac{1}{2}l_1W_1^2 + \ldots .) \\ a^2l^2(W^2 + Y^2) \qquad &= l_1^2(W_1^2 + Y_0^2)[1 - \tfrac{1}{2}l_1^2(W_1^2 + Y_0^2) + \ldots .] \end{aligned}\right\} \quad (33)$$

To the values of l_1, W_1, Y_0 for the reduced parallaxes (Table 9) correspond those of l, W, Y for the absolute magnitudes given in

TABLE 10

ADOPTED RESULTS: ABSOLUTE MAGNITUDES

Sp.	$\frac{W^2}{l(W^2+Y^2)}$	$\frac{1}{l}$	$1+\frac{Y^2}{W^2}$	$W=q$	$Y=\theta''$	θ	θ'
A1–F3	1.19	2.11	1.78	$\pm0^m.77$	$\pm0^m.68$	$\pm0^m.52$	$\pm0^m.51$
F4–F9	1.24	2.25	1.82	0.74	.66	.50	.49
G0–G7	1.44	2.47	1.72	0.78	.66	.53	.50
G8–M5	1.01	1.88	1.86	0.56	.52	.38	.38
G8–K2	1.05	2.70	2.56	0.66	.82	.51	.51
Main Sequence	1.18	2.22	1.88	0.69	.65	.48	.47
Giants	1.10	1.33	1.22	±1.02	±0.47	±0.44	±0.43

Table 10. The second, third, and fourth columns give the slopes of the regression line, the impartial line, and the recalibrated impartial line. The first of these has substantially the same value as that applicable to the parallaxes; the other two are slightly smaller.

23. The constant W is the average dispersion of the absolute magnitude from the mean value for its spectral subdivision (G0, G1, etc.) and is almost identical with that called q by Strömberg, which

is the dispersion about the mean for his wider groups, e.g., Go–G7. For this he finds $\pm 0\overset{m}{.}87$ for Go–G7 and $\pm 0\overset{m}{.}81$ for G8–K2. Our values are smaller, because in § 10 we have included determinations of W_1^2 (from $l_1W_1^2$ and $l_1^2[W_1^2 + Y_0^2]$, which, on the average, give smaller values). Had we confined ourselves to the column headed W_1^2 in Table 5, we would have found $W = \pm 0.79$ (Go–G7) and ± 0.74 (G8–K2).

24. The mean error of the spectroscopic absolute magnitudes is of special interest. It is the mean-square value of $M_s - M$, but may still be defined in several ways.

1. We may take M_s directly from *Mount Wilson Contribution* No. 511. The resulting mean error is called θ by Strömberg (B, p. 7).

2. We may modify the published values (or suppose them to have been corrected) by means of the regression lines defined in Table 10, thus obtaining spectroscopic parallaxes M_s' in agreement with the existing trigonometric parallaxes, provided that means are taken for stars grouped according to M_s'. The mean error of these absolute magnitudes may be called θ'.

3. We may correct the published values by means of the impartial lines, obtaining spectroscopic parallaxes M_s'', which agree with the trigonometric data for means of stars selected in some way which is not affected by the errors of either type of parallax. Call the mean error in this case θ''.

In the first case, we have, by equations (31), (32), and (33),

$$M_s - M = \bar{M}_s - \bar{M} + (l - 1)w + ly\,, \tag{34}$$

whence

$$\theta^2 = (\bar{M}_s - \bar{M})^2 + (l - 1)^2W^2 + l^2Y^2\,. \tag{35}$$

In the second case, we have

$$M_s' - \bar{M}_s = \frac{W^2}{l(W^2 + Y^2)}(M_s - \bar{M}_s)\,. \quad \text{(A, p. 4, eq. [8])}$$

The relations are formally as before, with $l' = W^2/(W^2 + Y^2)$ in place of l and

$$\theta'^2 = (\bar{M}_s - \bar{M})^2 + \frac{W^2Y^2}{(W^2 + Y^2)}\,.$$

In the third case,

$$M_s'' - \bar{M}_s = \frac{1}{l}(M_s - \bar{M}_s)$$

and

$$\theta''^2 = (\bar{M}_s - \bar{M})^2 + Y^2 .$$

The terms in $\bar{M}_s - \bar{M}$ are almost always negligible. The resulting values of θ, θ', and $\theta'' = Y$ are given in Table 10. For G0–G7, Strömberg's final values in C are $\theta = \pm 0.51$, $\theta'' = \pm 0.60$, and for G8–K2, $\theta = \pm 0.75$, $\theta'' = \pm 0.65$. The agreement in the first case is fairly good; in the second, the number of stars is too small to give accurate results.

The values of θ' are a little smaller than those of θ, while Y is much greater. If W and Y are constant in equation (35), the minimum of θ^2 occurs when $l = W^2/(W^2 + Y^2) = l'$ and, in general,

$$\theta^2 = \frac{W^2Y^2}{(W^2 + Y^2)} + (W^2 + Y^2)(l - l')^2 .$$

Of all the possible linear relations between $M_s - \bar{M}_s$ and $M - \bar{M}$, the regression line therefore gives the minimum "mean error" of $M_s - M$ for any large group selected in a *generally* impartial manner, without regard to the deviations of M or M_s. The results of this calibration will be systematically correct for the means of groups selected out of the assemblage by grouping by M_s, but systematically in error for groups selected by any impartial criterion.

In the great majority of cases in which spectroscopic parallaxes are used, the selection is impartial, or nearly so, with respect to the errors of these parallaxes. Hence, to obtain results which shall be systematically correct, we must adopt a calibration which considerably increases the apparent accidental errors of our parallaxes.

This has been proved for a linear correlation; but as the actual correlation is approximately linear for the great majority of the stars, it must also be true in practice.

If the values of l_1, W_1, Y_0 found for "All" in our earlier paper (or in Table 5) are treated as in § 22, we find $q = W = \pm 0.73$, $\theta'' = Y = 0.80$, $\theta = \pm 0.54$. Our present values are smaller on account of the

diminution of W_1 and Y_0 and increase of l_1, resulting from the inclusion of the results of the second method of discussion in Table 9. The agreement of the entirely independent results from the first four spectral groups in this table gives ground for confidence in the results. For the late-type dwarfs all the dispersions are significantly smaller than for the rest.

The only point of difference which remains between us and Dr. Strömberg is that he favors calibration by the impartial-relation line (§ 6, eq. [5]) when selection has been made "on a basis not *directly* connected with either M or M_s."

We are unable to concur with this conclusion, since this line results from the assumption that the method of grouping is correlated in equal measure with the deviations of M and M_s. It seems evident to us that any grouping that can be made without knowledge of the peculiarities of the individual spectra is *ipso facto* free from correlation with the deviations of M_s, whether arising from errors of observation or from physical peculiarities of the stars. The deviations in M, which are real properties of the stars, must appear in any group, however selected.

Our conclusion is therefore that unless it is *known* that the method of grouping is influenced by the intensities of the spectral lines, the calibration which we have called "impartial" should be adopted. This is equivalent to one by the second regression line of M_s upon M.

We recommend, therefore, that future lists of spectroscopic absolute magnitudes and parallaxes be calibrated in this way. The constants necessary to pass from this calibration to that by the first regression line (or to any intermediate grouping) should be given in the introduction to the catalogue.

25. Important evidence bearing on the dispersion of spectroscopic absolute magnitudes has been briefly presented by Adams, Joy, Humason, and Brayton.[17] For the components of a physical pair or a moving cluster, the modulus $m - M_s$ should be the same, and the residuals from the mean arise from the errors of m and M_s.

We are greatly indebted to Mr. Joy for putting at our disposal the original manuscript of the discussion, involving 157 double stars, 74 members of the Hyades, 31 of Praesepe, and 12 of h and χ Persei.

[17] *Mt. W. Contr.*, No. 511, p. 6; *Ap. J.*, **81**, 192, 1935.

If v denotes the residual difference of an observed $m - M_s$ from the mean for the system or cluster, and ϵ the mean error of $m - M_s$, we have $\epsilon^2 = \Sigma v^2/(N - n)$, where N is the number of stars included and n that of the mean values. It is evident that smaller values of ϵ are to be expected when m (or Δm for a double star) has been measured photometrically than when it has merely been estimated.

There are 64 stars of the Hyades for which photometric measures are available. For these the mean $m - M_s$ is 3.50, while for the 10 with estimated magnitudes it is 3.65. The value 3.5 has been adopted.

For Praesepe, very accurate magnitudes have recently been published by Haffner and Heckmann[18] corresponding to λ 6450 and λ 4270. The visual effective wave length may be taken as 5290, corresponding to a value of $1/\lambda$ three-sevenths of the way from the first to the second. Visual magnitudes obtained by a corresponding interpolation were compared with Joy's values of M, giving a mean $m - M_s = +5.86$.

Joy's values for h and χ Persei have been adopted. Three stars out of 12 were rejected by him as probably not cluster members. There are 80 double stars and one triple system for which photometric values of Δm are available. For each pair,

$$\Sigma v^2 = \tfrac{1}{2}(\Delta m - \Delta M_s)^2 \,.$$

The triple system was treated like a cluster.

The data for the various groups are summarized in Table 11, where r is the probable error corresponding to ϵ.

We have omitted the Perseus cluster from the final means, since it is peculiarly difficult to be certain whether the stars are really cluster members.

The mean values of ϵ from the photometric measures and estimates are—by accident—in exact agreement.

The high value of ϵ for the Praesepe group is remarkable. There is no doubt that the stars are really cluster members; their visual magnitudes are good, and the spectroscopic data should be com-

[18] *Veröff. Universitäts-Sternwarte Göttingen*, Nos. 53, 54, and 55, 1937.

parable with those for the brighter Hyades, which have the same spectral distribution. The actual dispersion in absolute magnitude for a given spectral class in Praesepe is the smallest known,[19] less than $\pm 0^m.05$. The difference might be explicable by the presence of numerous undiscovered spectroscopic binaries or unresolved physical pairs. In the list for Praesepe there are 6 binaries (19 per cent), and among the 64 well-observed Hyades stars, 31 (48 per cent) for

TABLE 11

RESULTS FOR CLUSTERS AND DOUBLE STARS

	N	n	Σv^2	ϵ^2	ϵ	r
	Photometric Magnitudes					
Hyades	64	1	8.29	0.132	$\pm 0^m.36$	$\pm 0^m.24$
Praesepe	31	1	11.37	.378	.62	.42
Double stars	161	80	17.05	.211	.46	.31
All	257	82	36.71	0.210	± 0.46	± 0.31
	Estimated Magnitudes					
h and χ Persei	9	1	4.23	0.529	$\pm 0^m.73$	$\pm 0^m.49$
Hyades	10	0*	3.08	.308	.56	.37
Double stars	152	76	14.70	.194	.44	.30
Last two	162	76	17.78	0.207	± 0.46	± 0.31

* Mean for the photometric m's used.

which the visual magnitudes have been corrected to the bright component. If these corrections are ignored, the mean value of ϵ for the Hyades is $\pm 0^m.41$, and the discordance persists.

Joy has rejected five visual pairs on account of great discordances. These are Boss 325, 1112, 2354, 2999, and 3531 with $\Delta m - \Delta M_s =$ -1.7, $+1.6$, -3.3, -2.3, and -2.7. The observations of all these stars are good, and they are all proved to be physical pairs by common proper motion. These five stars give $\Sigma v^2 = 14.46$. Added to Table 11, they give $\epsilon = \pm 0^m.62$ for the double stars and $\epsilon = \pm 0^m.53$

[19] Haffner and Heckmann, *Veröff. Universitäts-Sternwarte Göttingen*, No. 55, p. 89, 1937.

and $r = \pm 0^m.36$ for all stars observed photometrically. To exclude all five is rather drastic. The true value of ϵ must be near $\pm 0^m.5$.[20]

We have, for each star,

$$m = M + C + x + dm ,$$

where C is the mean modulus for the cluster, x allows for differences of parallax within it, and dm is the error of observation. Hence, by (34),

$$m - M_s = \text{const} + x + dm + (1 - l)w - ly$$

and, by (35),

$$\epsilon^2 = \overline{x^2} + \epsilon_m^2 + \theta^2 .$$

For photometric measures, ϵ_m is of the order of $\pm 0^m.1$; x is zero for double stars and negligible for Praesepe. For the Hyades, Smart's data[21] give for the 44 well-observed stars a mean-square distance $r = \pm 4.05$ parsecs from the center of the cluster, whose distance is $R = 35.5$ parsecs. For random distribution

$$x^2 = 25 \text{ Mod}^2 \cdot \frac{r^2}{3R^2}$$

and $x = \pm 0^m.14$, which corresponds to $x = \pm 0^m.07$ for the mean of the 257 stars. Hence, $\theta^2 = \epsilon^2 - 0.015$ and $\theta = \pm 0^m.44$ for these stars and $\pm 0^m.52$, including the five discordant cases. Of the 257 stars, 23 are giants. Hence, by Table 10, we should expect to find $\theta = \pm 0^m.48$. The agreement is satisfactory.

[20] A letter from Professor Joy states that the discordance for Boss 3531 is removed by a better spectrum of component B. For Boss 325 the spectrum of B is peculiar, and Boss 1112 is probably a subgiant. The discordances for the other two stars, ϵ Hydrae and ι Leonis, are real. If these alone are included, we find for the double stars, $\epsilon = \pm 0^m.55$; and for all stars observed photometrically, $\epsilon = \pm 0^m.50$, $r = \pm 0^m.34$, and $\theta = \pm 0^m.49$.

Our discussion was based on the published material and included all the discordant cases. Professor Joy believes that the rejection of the extreme discordances gives a more nearly correct determination of the general accuracy of the spectroscopic method. As these discordances must arise from physical causes in the stars, we do not oppose this view.

[21] *M.N.*, **99**, 176, 1939.

It is probable that the components of a double star, or of a cluster, are more similar in composition than the stars at large, and hence that W should be smaller for them. Adopting $V = \pm 0\overset{m}{.}64$, $l = 0.453$ from Table 10, we find, for the 257 stars, $W^2 = \pm 0\overset{m}{.}65$. For the Hyades, $\theta^2 = 0.102$ and $W = \pm 0\overset{m}{.}25$. Hopmann[22] finds $\pm 0\overset{m}{.}32$.

The results of this method of study are therefore fully concordant with the others.

26. In conclusion, it appears that the various investigations of the relation between spectroscopic and trigonometric parallaxes are fully consistent, within the limits set by differences in analytical method and by errors of sampling. The published calibration in *Mount Wilson Contribution* No. 511 is nearly correct for the mean of each spectral subclass but underestimates the differences in absolute magnitude among individual stars.

The calibration based on the regression line of M upon M_s (which it was the purpose of this catalogue to establish) may be improved, now that many more trigonometric parallaxes are available, by increasing these differences by 19 per cent for main-sequence stars and 10 per cent for giants.

A calibration based on the regression line of M_s upon M and giving correct absolute magnitudes for the mean of a group of stars selected at random demands the multiplication of the difference in the tabular absolute magnitude by 2.21 for main-sequence stars and 1.39 for giants.

This correction, including correction to the zero point, may be made by the formula[23]

$$M_s' = M_s + A(M_s - S_1) , \tag{36}$$

where M_s is the absolute magnitude published in *Mount Wilson Contribution* No. 511, M_s' the corrected value, $A = (1/l) - 1$, and S_1 is given in Table 12. For the main sequence, $A = 1.22$ and for giants, 0.33.

The values of S_1 have been recalculated with the equations given in A (pp. 29–30). For the giants, new values of s_0 and t_0, based on

[22] *A.N.*, **269**, 88, 1939.

[23] A, p. 30, eq. (42).

free-hand curves, have been substituted for the group means of paper A, Table 4.

It is doubtful whether the correction (36) should be applied to its full amount to stars for which $M_s - S_1$ exceeds about $1^m.5$, that is, to supergiants, subgiants, and subdwarfs. These objects really deserve special calibrations. Meanwhile, it may be desirable to use the smaller values given in A (cf. § 5).

TABLE 12

VALUES OF S_1

Main Sequence						Giants	
Sp.	S_1	Sp.	S_1	Sp.	S_1	Sp.	S_1
A1	1.7	F5	3.4	G9	5.3	G0–G4	+0.9
A2	1.7	F6	3.5	K0	5.5	G5	+ .8
A3	1.7	F7	3.6	K1	5.7	G6	+ .7
A4	1.7	F8	3.7	K2	6.0	G7	+ .6
A5	1.7	F9	3.8	K3	6.3	G8	+ .6
A6	1.7	G0	4.0	K4	6.6	G9	+ .5
A7	1.8	G1	4.1	K5	6.9	K0	+ .4
A8	2.1	G2	4.3	K6	7.3	K1	+ .3
A9	2.3	G3	4.4	M0	8.4	K2	+ .2
F0	2.6	G4	4.6	M1	8.9	K3	+ .1
F1	2.8	G5	4.7	M2	9.4	K4	.0
F2	3.0	G6	4.8	M3	9.9	K5	− .1
F3	3.2	G7	5.0	M4	10.6	M0	− .3
F4	3.3	G8	5.1	M5	11.6	M1, M2	− .4
						M3, M4	− .5
						M5	− .6
						M6	−0.7

27. The outstanding mean error θ'' of a spectroscopic parallax thus impartially calibrated is $\pm 0^m.64$ for main-sequence stars, not much smaller than the dispersion $W = \pm 0^m.69$ of the true absolute magnitudes about the mean for the spectral subclass. It would be quite wrong, however, to conclude that spectroscopic parallaxes were very little better than "spectral parallaxes" derived on the assumption that the absolute magnitude was equal to this group-mean; for these parallaxes are systematically in error and conceal the true differences in absolute magnitude, which the spectroscopic parallaxes reveal, though with some casual error.

It is curious that a biased calibration based on the regression

line, which removes but a small part of the systematic error, diminishes the "accidental" discordances so much that its results appear to have nearly twice the weight.

For the giants, $\theta'' = \pm 0^{m}.53$, $W = \pm 1^{m}.04$, and the superiority of the spectroscopic parallaxes is obvious.

The spectroscopic method, impartially calibrated, predicts the absolute magnitude of a main-sequence star with a probable error of $\pm 0^{m}.44$ and the parallax with a probable error of ± 21 per cent. For giants the probable errors are $\pm 0^{m}.32$ and ± 16 per cent. No other method of comparable accuracy is available except for the nearer stars.

It is a pleasure to express our thanks to Mr. Strömberg for many profitable discussions and for lending us copies of the manuscript of his recent paper, and to Mr. Joy for communicating the data discussed in § 25.

PRINCETON UNIVERSITY OBSERVATORY,
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April 9, 1940

THE LIGHT-CURVE OF ζ AURIGAE*

WILLIAM H. CHRISTIE

ABSTRACT

The light-curve of ζ Aurigae has been derived from photographic observations made with the 10-inch Cooke refractor combined with measures made elsewhere. The duration of the partial phase was 1.35 days, and that of totality 36.80 days. The middle of eclipse occurred J.D. 2429637.18.

The observations can be well represented, within the errors of observation, by a light-curve computed on the assumption that the B-type star is undarkened at the limb and that the limb of the K star is sharply defined.

The 1939–40 eclipse of ζ Aurigae was well observed at ingress but, unfortunately, during egress clouds interfered at most observatories. At Mount Wilson, however, some photometric observations were made during egress, despite haze and occasional clouds.

We have now enough material from which to determine the form of the light-curve with some degree of accuracy, thanks to the observations made at critical times by Brück and Green[1] at Cambridge, England; by Guthnick and Schneller[2] at Berlin-Babelsberg; and by Walter[3] at Potsdam. These observations have been combined with those of the Lick,[4] Mount Wilson, and Steward[5] observatories.

The Mount Wilson photographic observations were made with the Schraffierkassette attached to the 10-inch Cooke refractor. The magnitudes of the comparison stars were determined by direct comparisons with stars in the vicinity of the North Polar Sequence. Oosterhoff's[6] comparison stars are also included in this list, and the magnitudes of these stars have been revised. These stars are listed in Table 1 and should form a useful sequence for future observations.

Table 2 lists the Mount Wilson observations; the third column gives the number of exposures from which the magnitude was derived.

* *Contributions from the Mount Wilson Observatory, Carnegie Institution of Washington*, No. 635.

[1] *I.A.U. Circ.*, No. 807, 1940.

[2] *I.A.U. Circ.*, No. 808, 1940.

[3] *I.A.U. Circ.*, No. 808, 1940.

[4] Letter to the author.

[5] *Harvard Announcement Card*, No. 518, 1939.

[6] *Mt. W. Contr.*, No. 518; *Ap. J.*, **81**, 461, 1935.

All available observations made at ingress are plotted in Figure 1, and all have been reduced to an arbitrary scale because the amplitude of the observed light-variation is a function of the effective wave length at which the determinations were made. In combining observations made at different times, it is essential to remember that

TABLE 1

COMPARISON STARS FOR ζ AURIGAE

Star	m_{pg}	Star	m_{pg}
BD+41°1163	4.96	BD+40°1000	5.65*
BD+37 1005	5.00	BD+42 1881	5.73†
BD+38 1063	5.04	BD+42 1081	5.79*
BD+39 1248	5.29	BD+43 1116	5.99*†
BD+43 1043	5.32*	BD+40 1032	6.10*
BD+40 1268	5.52	BD+41 1045	6.18
BD+40 1253	5.57	BD+39 1135	6.27

* Oosterhoff's list. † Low weight.

TABLE 2

MAGNITUDES OF ζ AURIGAE

J.D. 2429000+	m_{pg}	No.	J.D. 2429000+	m_{pg}	No.
616.616	4.97	7	618.004	5.37	4
.679	4.96	4	.680	5.64	4
.958	5.09	4	.626	5.64	6
617.003	5.02	5	655.607	5.69	2
.610	5.04	8	.833	5.71	5
.727	5.16	6	656.636	5.15	13
.800	5.15	6	.736	5.12	8
.880	5.29	4	.851	5.09	6
.953	5.35	6	.903	5.06	1

there is a secondary, apparently irregular, variation with a range of nearly a tenth of a magnitude. This variation is attributed to the K star since it is observable when the B-type star is eclipsed. For this reason only observations made at ingress are included in the figure. Observations at egress are too meager to determine the zero point of the curve with accuracy.

The most astonishing result of this investigation is that the ob-

servations can be well represented by a light-curve computed on the assumption that the B star is undarkened at the limb and that the limb of the K star can be considered a straight occulting edge. This is wholly contrary to Menzel's[7] conclusions that "the B star sinks

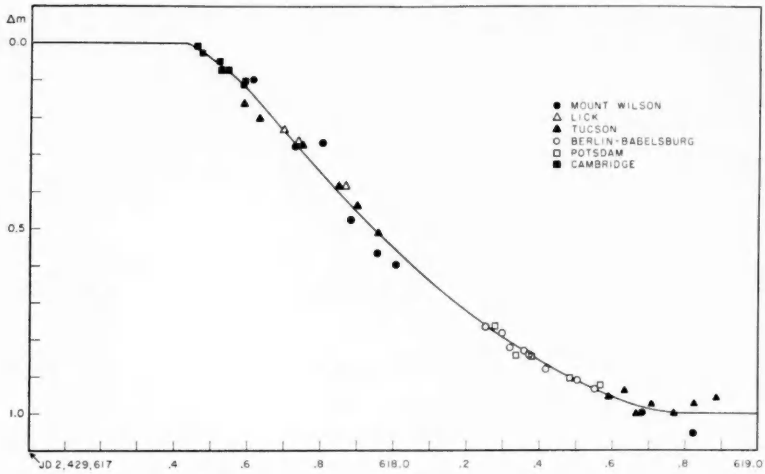

FIG. 1.—Light-curve of ζ Aurigae at ingress, December, 1939

TABLE 3

REVISED ELEMENTS OF ζ AURIGAE

Period	972.24 days
Epoch of minimum	J.D 2429637.18
Duration of eclipse	39.50 days
Duration of totality	36.80 days
Duration of partial phase	1.35 days
R_B	4.7×10^6 km $=6.75$ ⊙
ρ_B	0.027 ⊙

into obscurity behind the K like a planet setting in a smoky atmosphere, disappearing before it reaches the horizon."

The observations indicate, therefore, that general absorption of the light of the B star by that of the atmosphere of the K star is almost negligible at heights exceeding a small fraction of the diameter

[7] *Harvard Circ.*, No. 417, 1936.

of the B star—perhaps but a few tens of thousands of kilometers at the most.

New spectroscopic elements of the orbit of the binary derived by the writer show no important changes from the elements previously published.[8] The only revisions in the elements to be made at this time are given in Table 3.

In conclusion, I wish to express my thanks to those observers who so kindly sent their observations to me previous to publication. These include Dr. H. A. Brück and Mr. H. E. Green of Cambridge, England; Dr. C. M. Huffer of the Washburn Observatory; Dr. G. E. Kron of the Lick Observatory; Dr. F. E. Roach of the Steward Observatory; and Mr. E. G. Williams of the Radcliffe Observatory.

Carnegie Institution of Washington
Mount Wilson Observatory
May 1940

[8] Christie and Wilson, *Mt. W. Contr.*, No. 519; *Ap. J.*, **81**, 426, 1935.

SPECTROSCOPIC OBSERVATIONS OF BARNARD'S VARIABLE IN MESSIER 3*

ALFRED H. JOY

ABSTRACT

The velocity-curve of Barnard's variable star in the globular cluster M3 has been derived from 15 spectrograms. The normal velocity is −152.8 km/sec; the range, 56 km/sec.

The spectral type varies from F6 to G3. Emission lines of hydrogen occur during the increase in light but show no variation in position with period.

The spectral changes resemble those of the peculiar variable W Virginis.

The variability of one of the brighter stars near the center of the globular cluster Messier 3 was discovered by E. E. Barnard[1] in 1899. He found a visual range of 1.8 mag. and a period of 15.77594 days. More recently J. L. Greenstein[2] has deduced new elements from photographs of the cluster taken at Harvard and Mount Wilson. His light-curve closely resembles C. A. Chant's curve[3] of W Virginis, whose period of variation is 17.3 days. Barnard's variable is only 26″ from the center of the cluster; and, on account of the interference of neighboring stars, it is probable that the range of light-variation (12.9–14.0 pg. mag.) is somewhat underestimated.

Heretofore, few spectroscopic observations of individual stars in the globular clusters have been made. Because variables with periods greater than 1 day are seldom found in such systems, it is of especial interest to compare the behavior and physical characteristics of Barnard's star with Cepheids not in clusters.

The great scale (1 mm = $4\overset{\prime\prime}{.}7$) at the 135-foot Cassegrain focus of the 100-inch reflector permits easy identification and spectrographic observation of the brighter individual stars of clusters if the seeing is reasonably good. Spectrographic observations of Barnard's variable were begun in 1935, and a preliminary report on the spectrum and velocity changes was made in 1937.[4] The total

* *Contributions from the Mount Wilson Observatory, Carnegie Institution of Washington*, No. 637.

[1] *A.N.*, **172**, 345, 1906.

[2] *Harvard Bull.*, No. 901, 1935.

[3] *Harvard Ann.*, **80**, 225, 1917.

[4] *Pub. A.A.S.*, **9**, 45, 1937.

number of observations was later increased to fifteen by the addition of seven plates in 1938–1939. Study of the new material has made it necessary to revise some of the conclusions of the initial paper. The last two plates were made with the 9-inch camera (75 A/mm at $H\gamma$); all others with the 6-inch camera (120 A/mm). The data for the observations, the estimated spectral types, and the measured velocities, together with their assigned weights, are listed in Table 1.

TABLE 1

OBSERVATIONS OF BARNARD'S VARIABLE

Plate	JD	Phase	Spectrum	Velocity (Km/Sec)	Weight
C6678	2427945.764	0.121	F6	−180	1.0
6693	7964.767	.364	F8	163	1.0
6740	7975.736	.082	F5	165	0.7
6843	8226.924	.518	F8	150	1.0
6889	8269.876	.328	G0	180	1.0
6912	8291.792	.762	G0e	137	1.0
7013	8613.941	.842	G2e	128	0.3
7048	8673.779	.757	G3e	135	0.7
7167	9027.792	.921	G1e	139	1.0
7170	9055.780	.752	G2e	121	0.7
7252	9291.017	.145	F6	174	1.0
7261	9322.958	.235	F8	173	1.0
7273	9364.844	.975	G2e	134	1.0
7306	9411.781	.047	F6e	158	1.5
7328	9440.757	0.943	G0e	−141	1.5

The phases are decimal fractions of the period and are computed from Greenstein's elements in the form

$$\text{Maximum} = \text{JD } 2424627.55 + 15^{\text{d}}.28281 E .$$

The absorption lines are, in general, not so well defined as those in the spectra of galactic Cepheids. This is probably due to the presence of a faint background of light from other cluster stars lying nearly in the line of sight. There is a possibility that the velocity range is somewhat decreased on this account, but such an effect must be small.

The results of the measurements are plotted according to phase in Figure 1. In drawing the velocity-curve the weights of the points have been taken into account. The range of velocity is 56 km/sec;

and the normal velocity, determined by the method of equal areas, is -152.8 km/sec. The lag of maximum velocity of approach with respect to maximum light is 17 per cent of the period; and that of maximum velocity of recession with respect to minimum light, 20 per cent. The velocity-curve shows a general resemblance to those of the galactic Cepheids; but if the elements are correct, the lag of

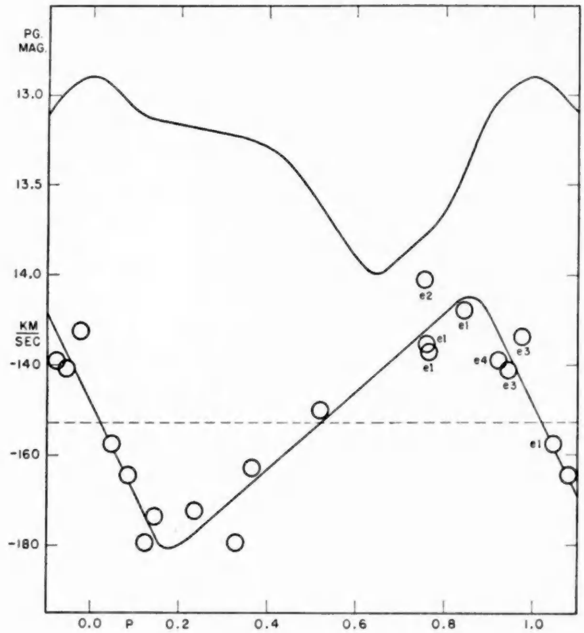

FIG. 1.—Velocity-curve of Barnard's variable in Messier 3. The presence and intensity of emission lines are indicated by the letter *e* and a numeral.

the velocity-curve of Barnard's star is considerably greater. For 14 Cepheids with periods between 13.6 and 17.3 days the average range of velocity is 55.7 km/sec, and the average lag is 12 per cent.

The variation in spectral type, as shown in Table 1, is from F6 at maximum to G3 at minimum, but the estimates are somewhat uncertain on account of the poor quality of the spectral lines. The average type is somewhat earlier than that of Cepheids having the same period. The star is evidently of high luminosity, as might be expected from the fact that it is one of the brightest in the cluster.

The most prominent lines in the spectrum are those of the ionized elements *Fe* II, *Sc* II, *Sr* II, and *Ti* II. If we use Greenstein's median photographic magnitude 13.4 and an absolute magnitude of -2.2 from the period-luminosity-curve, the modulus for the star is 15.6, which compares closely with the value of 15.43 for the cluster[5] and indicates that, in spite of its peculiarities, the absolute magnitude of Barnard's variable does not differ greatly from that of galactic Cepheids of the same period.

Emission lines of hydrogen appear on plates taken between minimum and maximum, when the light was increasing, and reach their greatest strength at phase 0.9, a short time before maximum light. The relative intensity of the emission lines on the different plates is indicated in Figure 1 by a numeral following *e*. The intensity of the lines decreases from $H\beta$ to $H\zeta$. The stronger lines are evidently made up of two components, of which the violet is more intense. As in W Virginis, measures of displacement of the emission lines $H\gamma$ and $H\delta$ show no certain variation of velocity with phase. The weighted mean velocity from the emission lines of seven plates is -194 ± 2.5 km/sec, which is a shift of 41 km/sec to the violet, compared with the normal velocity determined from the absorption lines, or of 54 km/sec, compared with absorption lines of the same plates.

The spectral changes, as well as the light-variations, correspond closely to those observed in W Virginis and differ distinctly from those of any other known variables. The galactic latitude of W Virginis ($+57°$) is anomalous for a galactic Cepheid, and the question may well be raised whether W Virginis may not have had its origin in a globular cluster and whether this type of variation may not be peculiar to the globular clusters.

Observations of several other bright variables in the globular clusters have been begun and will be carried on as opportunity permits.

CARNEGIE INSTITUTION OF WASHINGTON
MOUNT WILSON OBSERVATORY
June 1940

[5] Shapley and Sawyer, *Harvard Bull.*, No. 869, p. 6, 1929.

THE 48-INCH SCHMIDT TELESCOPE FOR THE ASTROPHYSICAL OBSERVATORY OF THE CALIFORNIA INSTITUTE OF TECHNOLOGY

FRANK E. ROSS

ABSTRACT

The color aberrations of the 48-inch Palomar Schmidt telescope have been computed for a number of wave lengths and have been combined in accordance with the spectral sensitivity range of various photographic emulsions. The variation of color aberrations with the arbitrary parameter k has been studied. A lens for flattening the field has been computed, and its color and field characteristics studied.

In designing a field-flattening lens for the 48-inch Schmidt telescope, which is in process of construction for the Palomar Observatory, it was necessary to investigate the outstanding color aberrations of the Schmidt correcting plate, both alone and in combination with the field-flattening lens. The spherical mirror, of diameter 72 inches, is now completed. Its radius of curvature is 241.125 inches. The computation will be made for Crystalex glass, which is similar in type to Schott K5.

The equation for a meridian section of a Schmidt correcting plate, assuming that one of the surfaces is flat, is, quite approximately,

$$x = \frac{-kr^2}{D} y^2 + \frac{1}{D} y^4 ; \qquad D = 4(n-1)R^3 , \tag{1}$$

where r is the radius of the correcting plate, R the radius of curvature of the mirror, n the index of refraction, k an arbitrary parameter, y the distance of any point of the plate from the axis, and x its distance from the vertex of the plate, measured along the axis. The color characteristics will be investigated for six cases with k between 0.5 and 2.0.

It will be of interest to give the value of x, the amount of glass removed, and the slope of the surface, S, for four cases. The results are given in Table 1. The slope, easily obtained from equation (1), is

$$S = \frac{2y}{D} (2y^2 - kr^2) , \tag{2}$$

and the ordinate for the neutral point, or the zone of zero slope,

$$y_n = \sqrt{\frac{k}{2}}\, r \, . \tag{3}$$

Computation of V.—Let V be the distance from the vertex of the mirror to the intersection of any ray with the axis. The vertex of the

TABLE 1

VALUES OF x AND S*

y (Inches)	$k=0.5$		$k=1.0$		$k=1.5$		$k=2.0$	
	x	S	x	S	x	S	x	S
4	− 1.5	− 3.9	− 3.0	− 8.3	− 4.5	−12.7	− 6.1	−17.1
8	− 4.8	− 4.9	−10.9	−13.7	−17.1	−22.5	− 23.3	−31.3
12	− 6.9	0.0	−20.8	−13.2	−34.6	−26.5	− 48.5	−39.7
16	− 2.7	+13.7	−27.3	− 3.9	−52.0	−21.6	− 76.5	−39.2
20	+15.0	+39.2	−23.5	+17.1	−62.0	− 4.9	−100.4	−26.9
22	+31.7	+57.2	− 4.9	+33.0	−61.7	+ 8.8	−108.0	−15.5
24	+55.4	+79.3	0.0	+52.9	−55.4	+26.5	−110.8	0.0

* The unit of x is 0.0001 inches; the unit of S is $0\overset{\circ}{.}001$.

correcting plate is assumed to coincide with the center of curvature of the mirror. The following formulae, easily derived, give V and the combined focal length, F:

$$\left.\begin{aligned} \sin\theta &= \frac{y}{R}\,, \\ B &= 2\theta + (n-1)S\,, \\ v &= \frac{R\sin\theta}{\sin B}\,, \\ V &= R - v\,, \\ F &= v \ (\text{for } y = y_n)\,. \end{aligned}\right\} \tag{4}$$

Since equation (1) is only an approximation, V will be found to vary with y. The maximum error of equation (1) is 0.0103 inches for $k = 0.5$, and the minimum is 0.0033 inches for $k = 2.0$. Before computing the color aberrations, these corrections have been applied.

Determination of the size and location of the images of different colors.—It is assumed that the plate has been figured for the wave length λ 4341, for which the spherical aberration will of course be zero. To the violet, the spherical aberration will be overcorrected; to the red, undercorrected. It is well known that the size and location of an axial image in which spherical aberration is present are determined from two rays, namely, the rim ray ($y = 24$ inches) and the two-thirds ray ($y = 16$ inches). Let Δ be the distance between the intersection points of these two rays with the axis, d the diameter of best image, and δV its location with respect to the intercept of the rim ray with the axis. We have

$$\left.\begin{aligned} \Delta &= V(r) - V(\tfrac{2}{3}r)\,, \\ d &= \frac{4}{5}\cdot\frac{r}{F}\,\Delta\,, \\ \delta V &= \tfrac{2}{5}\Delta\,. \end{aligned}\right\} \tag{5}$$

It is obvious that all the spherical aberration-curves will pass through a common point, which is the neutral point y_n corresponding to the value of k which is chosen. Now, if it is assumed that the spherical aberration-curves are linear from $y = \frac{2}{3}r$ to $y = r$, an assumption which is generally fulfilled, the best images, given by equation (5), will have a location given by the V corresponding to $y = a\cdot r$, where

$$a = 1 - \tfrac{2}{5}\cdot\tfrac{1}{3} = 0.867\,. \tag{6}$$

If, then, the neutral zone is located at $a\cdot r$, all the colored images will coincide. This is seen to be the case when $k = 1.5$, a result which is well known. Thus, $k = 1.5$ is a sufficient condition for minimum color-aberration. It will be shown, however, that it is not a necessary condition.

Table 2 has been computed by making use of equations (5) and gives the size and location of the colored images, for four wave lengths and six values of k. It is seen that, aside from computational differences, the image size is independent of k but that their location z depends strongly on k.

It will now be necessary to obtain d for the ranges in wave length corresponding to different types of emulsions. For the ordinary

blue-sensitive emulsion the spectral range can be taken from λ 3800 to λ 4861; for the panchromatic emulsion, without filter, from λ 3800 to λ 6563; for the panchromatic emulsion, with yellow filter, from λ 4861 to λ 6563. No general formula can be given for the computation. For the colors concerned it is necessary to plot both rim rays and rays at the two-thirds point in order to see which are

TABLE 2

VALUES OF d AND z

λ	k=0.50		k=1.00		k=1.25		k=1.50		k=1.75		k=2.00	
	d	z	d	z	d	z	d	z	d	z	d	z
		mm		mm		mm		mm		mm		mm
3800	1″35	+0.12	1″79	+0.06	1″36	+0.03	1″70	0.00	1″38	−0.02	1″77	−0.06
4341	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
4861	0.89	−0.07	0.79	−0.03	0.79	−0.01	0.81	0.00	0.85	+0.02	0.87	+0.09
6563	2.26	−0.20	2.28	−0.10	2.26	−0.05	2.28	0.00	2.28	+0.07	2.28	+0.10

TABLE 3

COMBINED IMAGE SIZE
(Scale, 10″=0.149 mm)

k	λ 3800–λ 4861	λ 4861–λ 6563	λ 3800–λ 6563
0.50	3″6	2″8	6″0
1.00	2.5	2.3	4.2
1.25	1.7	2.3	3.0
1.50	1.4	2.3	2.3
1.75	1.4	2.3	2.3
2.00	2.1	2.3	3.0

the effective rays governing the size of the image. When the colors are well separated, it will be found that the rim rays alone determine the image diameter. The results are contained in Table 3.

It can be concluded from Table 3 that k can have any value between 1.50 and 1.75 without affecting the size of the colored image. As far as I am aware, this has not heretofore been pointed out. It remains to be seen how this result is affected when a field flattener is included in the optical system. This will now be considered.

THE FIELD FLATTENER

With a Schmidt telescope the field is imaged on the surface of a sphere of radius equal to the focal length. In our case

$$R_s = F = 121.023 \text{ inches (for } k = 1.5) .$$

It was at first thought that the ordinary photographic plate could be bent to this curvature, but subsequent experiments did not confirm this, although the outlook for ultimate success is favorable. It is not certain that photographic film can be successfully used, especially for long exposures, on account of the large area involved. It is easy to show that a simple lens placed near the focus will flatten the field or change its curvature any desired amount. For a flat field in any optical system it is necessary only to satisfy the Petzval condition, namely,

$$\sum \frac{1}{nF} = 0 . \tag{7}$$

For the three optical elements,

$$F \text{ (mirror)} = 120.562 \text{ inches} , \qquad n = -1 .$$

The focal length of the Schmidt plate, easily derived, is

$$F_s = \frac{2R^3}{kr^2} = 32,452 \text{ inches} , \qquad n = 1.5341 .$$

Assuming the same glass for the flattener, we then have, from equation (7),

$$F \text{ (flattener)} = 78.779 \text{ inches} .$$

Assuming one surface flat, its radius of curvature is

$$R_1 \text{ (flattener)} = (n - 1) F = 42.076 \text{ inches} .$$

The computations will be made on the assumption that the flat surface is adjacent to the film, at a distance as small as is practicable. The diameter of the lens was chosen 20 inches, requiring a central thickness equal to 1.40 inches. It was found that if the vertex of the curved surface R_1 is placed 0.96 inches in front of the focus for the

Schmidt itself, the desired position of the image plane is secured. The color will now be investigated.

Color, Schmidt plate plus field flattener.—Assuming the lens to be placed as indicated (u = 0.96 inches for λ 4341), the color rays have been traced through the lens for five values of k. The results are contained in Table 4. The diameter of the image is d, and z is its distance from the flat surface of the lens.

TABLE 4

COMBINED COLOR; PLATE AND FLATTENER

k	λ 3800		λ 4341		λ 4861		λ 6563		RANGE
	d	z	d	z	d	z	d	z	z
		mm		mm		mm		mm	mm
1.00....	2″.97	1.26	1″.59	1.08	0″.76	0.98	0″.67	0.79	0.47
1.25....	2.97	1.23	1.52	1.08	0.79	0.99	0.60	0.83	0.40
1.50....	2.95	1.19	1.54	1.08	0.76	1.02	0.67	0.89	0.30
1.75....	2.97	1.16	1.52	1.08	0.76	1.03	0.65	0.94	0.22
2.00....	2.95	1.13	1.54	1.08	0.70	1.05	0.65	0.98	0.15

TABLE 5

COMBINED IMAGE SIZE; PLATE WITH FLATTENER

(Scale, 10″=0.149 mm)

k	λ 3800–λ 4861	λ 4861–λ 6563	λ 3800–λ 6563
1.00.............	5″.0	3″.3	8″.3
1.25.............	4.3	2.9	7.1
1.50.............	3.8	2.5	6.2
1.75.............	3.3	2.0	5.1
2.00.............	3.0	1.7	4.2

Table 5 can now be formed in a manner similar to the formation of Table 3 from Table 2.

As in the case of the Schmidt plate alone, Table 4 shows that the size of the colored images is independent of k. But when different colors are combined, the range of these colors along the axis, given

in column 10 of Table 4, must be taken into account. The range is seen to diminish with increasing k, so that the most favorable color conditions are obtained with k equal to 2.0. But a reference to Table 3, giving the size of images for the Schmidt plate alone, shows that the case $k = 2.0$ is distinctly unfavorable. The best compromise is seen to be $k = 1.75$. This gives a minimum color aberration for the Schmidt plate alone and is but little inferior to $k = 2.0$ for combined plate and flattener.

It must not be assumed that the size of threshold images is no less than the values of d, the diameter of the color disks, in Tables 3 and 5. This would be the case only if the stellar energy-curves, combined with the photographic sensitivity, were quite constant over the spectral ranges in question. This applies in particular to the radiation in the neighborhood of λ 3800, which has strongly influenced the calculation for d, but is comparatively weak in photographic efficiency.

Field characteristics.—It has been shown (p. 402) that for field-angle zero the location of the image is determined by the intersection with the axis of the rays at $y = 0.866\ r$. Computation shows that for λ 4341 this point is located 0.25 mm beyond the Gaussian focal plane. A similar calculation has been carried through for a field angle of 4°0, with the result that the meridian rays from this zone intersect at this same distance beyond the Gauss plane, but at a point 0.04 mm below the ray $y = 0$, thus showing a small amount of negative coma. A small amount of astigmatism is indicated. The effect of changing the form of the flattener without changing its power has been studied. Thus, the field characteristics were studied for $R = 48.8$ and 53.1, accordingly introducing curvature into R_2. It was found that the astigmatism was reduced, but at the expense of coma and of field flatness, discussed above.

Distortion.—The value of the distortion has been obtained by triangulating chief rays at angles of 2° and 4° through the field flattener and comparing the intercepts p on the Gaussian plane with the values computed from the simple relation

$$p_0 = F \tan \vartheta ,$$

where F is the Gaussian focal length and ϑ the field angle. Computation gives for the focal length of the optical system

$$F(G') = 119.556 \text{ inches} ; \qquad F(C) = 119.560 \text{ inches} .$$

Table 6 gives the results.

TABLE 6

DISTORTION AND FIELD COLOR

ϑ	G' (4341)			C (6563)	
	p (inches)	p_0 (inches)	D (inches)	p (inches)	Δp (inches)
2°0	4.1799	4.1750	+0.0049	4.1804	+0.0005
4.0	8.3996	8.3602	+0.0394	8.4001	+0.0005

The distortion is given in the fourth column of Table 6. As was to be expected, it is quite accurately proportional to the cube of the field angle, fitting the formula,

$$D \text{ (mm)} = +0.0156\, \vartheta^3 .$$

The amount of the color magnification difference is given in the last column; it is satisfactorily small.

Summary.—It would appear from this investigation that the choice of aperture and of focal length for the Schmidt telescope for the Palomar Observatory is a happy one. Used without the field flattener, the outstanding color aberrations are well within the tolerance of good photography. With the field flattener, the color aberrations are increased but not to the point of being objectionable. If necessary, better optical performance can be secured by reducing the power of the flattener, the residual curvature of the field being taken up by bending the plate.

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May 15, 1940

PHYSICAL PROCESSES IN GASEOUS NEBULAE

X. COLLISIONAL EXCITATION OF NEBULIUM

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ABSTRACT

Target areas for collisional excitation of *O* III by electron impact are calculated by wave-mechanical methods. Coulomb wave functions are used for the colliding electron. The resulting target areas are large, particularly for excitation between the levels of ^3P. Application to the nebulae leads to the result that collisional de-excitations from $^3P_2$ predominate over the spontaneous transitions. In consequence, the levels of $^3P_2$ attain a high population, suited to the mechanism proposed by Bowen: resonance absorption of λ 303 of *He* II by the coincident line $^3P_2 - {}^3P_2$ of *O* III. The figures also suggest, for the lower electron densities of interstellar space, a high population of atoms in the ground level, a result consistent with Dunham's observations of the interstellar lines of *Ti* II.

The high intensities of the forbidden lines in gaseous nebulae lead one to conclude that the mode of excitation for these lines may differ from that for normal lines. Bowen has suggested that the free electrons, liberated from hydrogen by photo-ionization, may excite the "nebulium" lines by inelastic impact. To calculate the effectiveness of this excitation process, we must have advance knowledge of the atomic collision cross-sections. Since experimentally determined values of these parameters are not available, we must have recourse to theoretical methods, via wave mechanics. The calculations of the target areas have been carried out specifically for the various metastable levels of *O* III, the atom responsible for the most intense of the nebular lines.

THE BORN APPROXIMATION

We consider an *O* III ion initially in the state determined by the quantum numbers J, L, S, M, which give, respectively, the total, orbital, and spin angular momenta and the component of the total angular momentum of the ion along some convenient axis. Let us suppose that an electron with momentum $\boldsymbol{p}$ at infinity and spin component m_s impinges on the ion, changing its state to J', L', S', M', while at the same time the electron escapes with a momentum $\boldsymbol{p}'$

within solid angle $d\Omega$ and spin component m_s'. The first-order differential cross-section for this process can be written[1]

$$\sigma(JLSM\mathbf{p}m_s \rightarrow J'L'S'M'\mathbf{p}'m_s')d\Omega = \frac{d\Omega}{4\pi^2}\frac{p'}{p}$$
$$(JLSM\mathbf{p}m_s|V|J'L'S'M'\mathbf{p}'m_s)(J'L'S'M'\mathbf{p}'m_s'|V|JLSM\mathbf{p}m_s)\ ,$$

or, briefly,

$$\sigma(n \rightarrow n')d\Omega = \frac{d\Omega}{4\pi^2}\frac{p'}{p}(n|V|n')(n'|V|n)\ . \qquad (1)$$

The bracketed symbol on the right side is the matrix element of V, the mutual potential of the electron and ion connecting the two complete states n and n'. We can write, symbolically,

$$(n'|V|n) = \int \Psi_{n'}^* V \Psi_n d\tau\ , \qquad (2)$$

where $\Psi_{n'}$ and Ψ_n represent wave functions for the complete system of free electron and ion, and the integral sign represents an integration over the space co-ordinates of the several electrons concerned. Throughout this paper an asterisk will be used to designate the complex conjugate. Disregarding the exclusion principle for the moment, we can write the complete wave function as the product $\Psi_n = \Psi_n\chi_n$, where Ψ_n refers to the ion alone and χ_n to the free electron. The usual procedure is to take plane waves for the free electron, i.e., to let $\chi_n = e^{i(\mathbf{p}\cdot\mathbf{r})}$. This procedure is suitable for collision problems involving neutral atoms, but for collisions with charged ions the incoming and outgoing waves will be greatly distorted from plane, especially for low velocities in which we are particularly interested. Instead of plane waves we shall therefore use functions that are exact solutions of the wave equation for the Coulomb field and which will be given later. The simple product functions $\Psi_n\chi_n$ do not

[1] See, e.g., Dirac, *Quantum Mechanics* (2d ed.), chap. ix, or Mott and Massey, *Theory of Atomic Collisions*, chap. viii. In eq. (1) and in the following we use *atomic units* in which the units of mass, charge, and action are, respectively, the electronic mass m and charge e (in absolute value) and Dirac's action constant $\hbar = h/2\pi$. The unit of length is then the first Bohr radius, and the unit of energy is twice the ionization energy of the hydrogen atom. For a complete list see Condon and Shortley, *Theory of Atomic Spectra*, p. 432. In eq. (1) we take $(n|V|n')(n'|V|n)$ instead of $|(n|V|n')|^2$, since we wish to have $p^2\sigma(n\rightarrow n') = p'^2\sigma(n'\rightarrow n)$. Cf. Mott and Massey, *op. cit.*, p. 153.

satisfy the exclusion principle and do not allow for the possibility of exchange between the incident electron and the electrons of the ion. To satisfy these two conditions, we must write Ψ_n as a sum of terms $\Psi_n\chi_n$, in which the arguments of the electrons are permuted. In our case we shall consider only the two 2p electrons of the ion, so that $\Psi_n = \Psi_n\,(1, 2)$ contains the co-ordinates of the two electrons. We shall suppose that Ψ_n already satisfies the exclusion principle, i.e., $\Psi_n(2, 1) = -\Psi_n(1, 2)$. Then the wave function for the complete system is given by

$$\sqrt{3}\Psi_n(1, 2, 3) = \Psi_n(1, 2)\chi_n(3) + \Psi_n(2, 3)\chi_n(1) + \Psi_n(3, 1)\chi_n(2)\,, \quad (3)$$

where the arguments $1, 2, 3$ stand for the co-ordinates and spin of the three electrons.

The potential V can now be stated explicitly, namely,

$$V(a, b, c) = \frac{2}{r_c} - \frac{1}{r_{ac}} - \frac{1}{r_{bc}} = V(a, c) + V(b, c)\,, \quad (4)$$

where r_c is the distance of the colliding electron from the ion, and r_{ac}, r_{bc} are the distances from the colliding electron to the two electrons of the ion. It should be noticed that V reduces to zero faster than $-2/r_c$ as r_c goes to infinity, since the Coulomb term $-2/r_c$ has been included in the wave equation for χ_n and V represents the actual mutual energy of the colliding electron and ion less this Coulomb term. If we insert the expressions (3) and (4) into (2), we obtain after some simplification

$$\left.\begin{aligned}(n'|V|n) = {} & 2\int\Psi^*_{n'}(1, 2)\Psi_n(1, 2)V(1, 3)\chi^*_{n'}(3)\chi_n(3)d\tau \\ & + 2\int\Psi^*_{n'}(2, 3)\Psi_n(1, 2)V(1, 3)\chi^*_{n'}(1)\chi_n(3)d\tau \\ & + 2\int\Psi^*_{n'}(3, 1)\Psi_n(1, 2)V(1, 3)\chi^*_{n'}(2)\chi_n(3)d\tau\,.\end{aligned}\right\} \quad (5)$$

The first term alone is what would be obtained with the simple product functions which neglect exchange. The second and third terms are clearly exchange integrals, the third being of a rather peculiar nature, since the potential $V\,(1, 3)$ does not directly connect the incoming and outgoing electrons 2 and 3.

We are not primarily interested in the differential cross-section (1)

for polarized beams of electrons and definite transitions of the magnetic quantum number M. Instead, we require the total integral cross-section $\sigma(JLSp \rightarrow J'L'S'p')$, which is obtained by integrating (1) over $d\Omega$, i.e., over all directions of p', averaging over the initial values M and m_s and summing over the final values M' and m_s'. The result is

$$\sigma(JLSp \rightarrow J'L'S'p') = \frac{1}{2(2J+1)} \sum_{MM'm_s m_s'} \int \sigma(n \rightarrow n') d\Omega \, . \tag{6}$$

We shall now examine the wave functions for the O III ion and for the free electron which enter into the integrals in equation (5).

WAVE FUNCTIONS FOR O III

The radial wave function for a 2p electron in O III has been given in tabular form by Hartree and Black.[2] It is convenient for our purpose to obtain an analytic approximation to this function in the manner proposed by Slater.[3] We write, namely, the radial function in the form

$$R(r) = r^2(ae^{-br} + ce^{-dr}) \, , \tag{7}$$

where r is the distance from the center of the ion, and adjust the constants a, b, c, d to give the best fit with the tabulated functions, subject to the normalization condition $\int_0^\infty R^2 dr = 1$. The adjusted values are

$$a = 10.415 \, , \qquad b = 3.604 \, , \qquad c = 4.520 \, , \qquad d = 2.014 \, .$$

The two-electron wave functions Ψ_{JLSM} (1, 2) of the ion are now built up from angular-momentum considerations.[4] The atom of O III is sufficiently near LS coupling for the ground configuration that we need not concern ourselves with the departures from precise LS coupling that are so important for optical transitions. We choose a spherical co-ordinate system (r, θ, ϕ) and let the polar axis $\theta = 0$ coincide with the direction of the momentum p of the incident electron. Then, if this axis is also the direction of quantization for the magnetic quantum numbers, we find

[2] *Proc. R. Soc.*, A, **139**, 311, 1933. [3] *Phys. Rev.*, **42**, 33, 1932.

[4] M. H. Johnson, *Phys. Rev.*, **39**, 197, 1932; Condon and Shortley, *op. cit.*, p. 220.

$$
\left.\begin{array}{cccccc}
\Psi_{JLSM}\,(1,2)r_1r_2/R(r_1)R(r_2) & J & L & S & M \\
\Phi_1\alpha_1\alpha_2 & 2 & 1 & 1 & +2 \\
\dfrac{1}{\sqrt{2}}\left[\Phi_1\,\dfrac{\alpha_1\beta_2+\beta_1\alpha_2}{\sqrt{2}} - i\Phi_0\alpha_1\alpha_2\right] & 2 & 1 & 1 & +1 \\
\dfrac{1}{\sqrt{6}}\left[\Phi_1\beta_1\beta_2 + \Phi_1^*\alpha_1\alpha_2 - 2i\Phi_0\,\dfrac{\alpha_1\beta_2+\beta_1\alpha_2}{\sqrt{2}}\right] & 2 & 1 & 1 & 0 \\
\dfrac{1}{\sqrt{2}}\left[\Phi_1^*\,\dfrac{\alpha_1\beta_2+\beta_1\alpha_2}{\sqrt{2}} - i\Phi_0\beta_1\beta_2\right] & 2 & 1 & 1 & -1 \\
\Phi_1^*\beta_1\beta_2 & 2 & 1 & 1 & -2 \\
\dfrac{1}{\sqrt{2}}\left[\Phi_1\,\dfrac{\alpha_1\beta_2+\beta_1\alpha_2}{\sqrt{2}} + i\Phi_0\alpha_1\alpha_2\right] & 1 & 1 & 1 & +1 \\
\dfrac{1}{\sqrt{2}}\left[\Phi_1\beta_1\beta_2 - \Phi_1^*\alpha_1\alpha_2\right] & 1 & 1 & 1 & 0 \\
\dfrac{1}{\sqrt{2}}\left[\Phi_1^*\,\dfrac{\alpha_1\beta_2+\beta_1\alpha_2}{\sqrt{2}} + i\Phi_0\beta_1\beta_2\right] & 1 & 1 & 1 & -1 \\
\dfrac{1}{\sqrt{3}}\left[\Phi_1\beta_1\beta_2 + \Phi_1^*\alpha_1\alpha_2 + i\Phi_0\,\dfrac{\alpha_1\beta_2+\beta_1\alpha_2}{\sqrt{2}}\right] & 0 & 1 & 1 & 0 \\
\Phi_{D2}\,\dfrac{\alpha_1\beta_2-\beta_1\alpha_2}{\sqrt{2}} & 2 & 2 & 0 & +2 \\
\Phi_{D1}\,\dfrac{\alpha_1\beta_2-\beta_1\alpha_2}{\sqrt{2}} & 2 & 2 & 0 & +1 \\
\Phi_{D0}\,\dfrac{\alpha_1\beta_2-\beta_1\alpha_2}{\sqrt{2}} & 2 & 2 & 0 & 0 \\
\Phi_{D1}^*\,\dfrac{\alpha_1\beta_2-\beta_1\alpha_2}{\sqrt{2}} & 2 & 2 & 0 & -1 \\
\Phi_{D2}^*\,\dfrac{\alpha_1\beta_2-\beta_1\alpha_2}{\sqrt{2}} & 2 & 2 & 0 & -2 \\
\Phi_S\,\dfrac{\alpha_1\beta_2-\beta_1\alpha_2}{\sqrt{2}} & 0 & 0 & 0 & 0
\end{array}\right\} \quad (8)
$$

where

$$\Phi_1 = \frac{3}{8\pi}[\cos\theta_1 \sin\theta_2 e^{i\phi_2} - \cos\theta_2 \sin\theta_1 e^{i\phi_1}]$$

$$\Phi_0 = \frac{3\sqrt{2}}{8\pi}\sin\theta_1 \sin\theta_2 \sin(\phi_1 - \phi_2)$$

$$\Phi_{D0} = \frac{\sqrt{6}}{8\pi}[\sin\theta_1 \sin\theta_2 \cos(\phi_1 - \phi_2) - 2\cos\theta_1 \cos\theta_2]$$

$$\Phi_{D1} = \frac{3}{8\pi}[\cos\theta_1 \sin\theta_2 e^{i\phi_2} + \cos\theta_2 \sin\theta_1 e^{i\phi_1}]$$

$$\Phi_{D2} = \frac{3}{8\pi}\sin\theta_1 \sin\theta_2 e^{i(\phi_1+\phi_2)}$$

$$\Phi_S = \frac{\sqrt{3}}{4\pi}[\cos\theta_1 \cos\theta_2 + \sin\theta_1 \sin\theta_2 \cos(\phi_1 - \phi_2)]\,.$$

In equations (8), α_i, β_i represent single-electron spin symbols for which the spin component is, respectively, $\pm\frac{1}{2}$. They satisfy $\alpha_i\beta_i = \beta_i\alpha_i = 0$ and $\alpha_i^2 = \beta_i^2 = 1$.

CONTINUOUS COULOMB WAVE FUNCTIONS

As already stated, the wave functions that we use to represent the incoming and outgoing electrons are the appropriate solutions of the wave equation in the Coulomb field of the ion. For the incident electron this function is[5]

$$\chi_{\boldsymbol{p},\pm 1/2}(3) = e^{\pi Z/2p}\Gamma\left(1 - \frac{iZ}{p}\right)e^{i(\boldsymbol{p}\cdot\boldsymbol{r}_3)}{}_1F_1\left(\frac{iZ}{p};\ 1;\ ipr_3 - i(\boldsymbol{p}\cdot\boldsymbol{r}_3)\right)_{\beta_3}^{\alpha_3}, \quad (9)$$

where Z is the charge on the ion (in our case $Z = 2$), and ${}_1F_1$ is a confluent hypergeometric function. Similarly, the wave function for the scattered electron is

$$\chi_{\boldsymbol{p}',\pm 1/2}(3) = e^{\pi Z/2p'}\Gamma\left(1 + \frac{iZ}{p'}\right)e^{i(\boldsymbol{p}'\cdot\boldsymbol{r}_3)}{}_1F_1\left(-\frac{iZ}{p'};\ 1;\ -ip'r_3 - i(\boldsymbol{p}'\cdot\boldsymbol{r}_3)\right)_{\beta_3}^{\alpha_3}. \quad (10)$$

The function (9) reduces approximately to a plane wave of unit amplitude at large distances in the direction opposite to $\boldsymbol{p}$, while

[5] Mott and Massey, *op. cit.*, chap. iii.

(10) is asymptotic to an approximate plane wave in the forward direction of $\boldsymbol{p}'$. For $Z = 0$, (9) and (10) become, respectively, $e^{i(\boldsymbol{p}\cdot\boldsymbol{r}_3)}$ and $e^{i(\boldsymbol{p}'\cdot\boldsymbol{r}_3)}$, apart from the spin factors α_3 and β_3.

In the next section we shall require the expansions of (9) and (10) in surface harmonics. With the co-ordinates defined in the last section, (9) is the same as[6]

$$\chi_{\boldsymbol{p},\pm 1/2}(3) = \frac{1}{r_3}\sqrt{\frac{2}{p}}\sum_{l=0}^{\infty}\sqrt{(2l+1)}\, i^l e^{i\eta_l} F_l(r_3) P_{l0}\,(\cos\theta_3)^{\alpha_3}_{\beta_3}, \tag{11}$$

where

$$F_l(r_3) = \frac{e^{\pi Z/2p}}{2\sqrt{p}}\cdot\frac{\left|\Gamma\left(l+1-\frac{iZ}{p}\right)\right|}{(2l+1)!}\, e^{ipr_3}(2pr_3)^{l+1}\,{}_1F_1\left(l+1-\frac{iZ}{p}\,;\,2l+2;\,-2ipr_3\right), \tag{12}$$

and P_{lm} is the normalized Legendre function which satisfies $\int_0^{\pi} P^2_{lm}\,(\cos\theta)\sin\theta d\theta = 1$. The phases η_l are functions of p, but their precise form is of no particular interest in the present work. It can easily be shown that the F_l defined by (12) are real. The expansion of (10) is similar to (11), since (10) can be obtained from (9) by reversing the sign of i and replacing $\boldsymbol{p}$ by $-\boldsymbol{p}'$. A difference in form arises, since $\boldsymbol{p}'$ is not directed along the polar axis as is $\boldsymbol{p}$. If we let the direction of $\boldsymbol{p}'$ be specified by θ, ϕ, and apply the addition theorem[7] for Legendre polynomials, we find

$$\left.\begin{aligned}&\chi_{\boldsymbol{p}',\pm 1/2}(3)\\ &= \frac{1}{r_3}\sqrt{\frac{2}{p'}}\sum_{l'=0}^{\infty}\sqrt{(2l'+1)}\, i^{l'} e^{i\eta'_{l'}} F'_{l'}(r_3)\sum_{m'=-l'}^{l'} P_{l'm'}\,(\cos\theta_3) P_{l'm'}\,(\cos\theta) e^{im'(\phi_3-\phi)\alpha_3}_{\quad\beta_3}\,.\end{aligned}\right\} \tag{13}$$

Our functions $F_l(r)$ differ slightly from those usually defined. For example, the quantity $L_l(r)$ defined on page 39 of Mott and Massey's book[1] is, except for notation and units, the same as $F_l(r)/(r\sqrt{p})$.

[6] C. G. Darwin, *Proc. R. Soc.*, A, **118,** 654, 1928; H. A. Bethe, *Handb. d. Phy.* (2), **24,** Part I, p. 554.

[7] Whittaker and Watson, *Modern Analysis*, p. 326.

The reason for our choice[8] is that for small p, $F_l(r)$ is rather insensitive to p near the origin $r = 0$, while under the same conditions $L_l(r)$ varies as $1/\sqrt{p}$. This means that, using $F_l(r)$, we can express most of the dependence of the cross-sections on the velocity explicitly, in the region of small velocities, and the implicit dependence entering through the F_l is less important.

METHOD OF PARTIAL CROSS-SECTIONS

Each of the matrix elements (5), which must be evaluated to obtain the cross-sections, contains integrals over the nine space coordinates of the three electrons.[9] In order to perform these integrations, we use the method of partial cross-sections, i.e., we expand the wave functions of the incoming and outgoing electrons and the potential function in surface harmonics and then integrate term by term. For neutral atoms it is well known that the successive terms in this method decrease rapidly for small velocities on account of the fact that for such velocities the classical distance of closest approach increases rapidly with the angular momenta l, l'. With ions instead of neutral atoms, of course this convergence will not be quite so good. The required expansions for the continuous wave functions have already been given in equations (11) and (13). For V we have

$$\left.\begin{aligned} V(r_1, r_3) &= \frac{1}{r_3} - \frac{1}{r_{13}} \\ &= -\sum_{s=0}^{\infty} \gamma_s(r_1, r_3) \sum_{t=-s}^{s} P_{st}(\cos\theta_1) P_{st}(\cos\theta_3) e^{it(\phi_1 - \phi_3)} \end{aligned}\right\} \tag{14}$$

with

$$\begin{aligned} \gamma_s(r_1, r_3) &= \frac{2}{2S+1}\left(\frac{r_1^s}{r_3^{s+1}} - \frac{\delta_s^0}{r_3}\right) && (r_1 < r_3) \\ &= \frac{2}{2S+1}\left(\frac{r_3^s}{r_1^{s+1}} - \frac{\delta_s^0}{r_3}\right). && (r_1 > r_3) \end{aligned}$$

If we interchange the order of summation and integration, we have a multiple sum of integrals corresponding to each integral of (5), the

[8] Yost, Wheeler, and Breit, *Phys. Rev.*, **49**, 174, 1936.

[9] The spin symbols disappear on application of the rules given after eqs. (8).

indices of summation being l, l', m', s, t. This result is appreciably simplified when the angular integrations are considered. The integrations over the angles ϕ_1, ϕ_2, ϕ_3 and most of those over θ_1, θ_2, θ_3 are elementary. A few integrals occur that contain the products of three associated Legendre functions, and for these the rules of Gaunt[10] are useful. As is to be expected, the angular integrations severely restrict the indices l, l', m', s, t. Before specifying the relations between these indices, we find it convenient to distinguish the three types of integral arising from (5). It will be seen that the threefold radial integrals which remain after the angular integrations have been carried out are of the three following forms, corresponding to the three terms in (5):

$$f_{l'l}^{(s)}(p';p) = \iiint dr_1 dr_2 dr_3 R_1^2 R_2^2 \gamma_s(r_1, r_3) F'_{l'}(r_3) F_l(r_3) \,, \tag{15}$$

$$g_{l'l}^{(s)}(p';p) = \iiint dr_1 dr_2 dr_3 R_1 R_2^2 R_3 \gamma_s(r_1, r_3) F'_{l'}(r_1) F_l(r_3) \,, \tag{16}$$

$$h_{l'l}^{(s)}(p';p) = \iiint dr_1 dr_2 dr_3 R_1^2 R_2 R_3 \gamma_s(r_1, r_3) F'_{l'}(r_2) F_l(r_3) \,. \tag{17}$$

Then, on account of the angular integrations, only the following terms appear:

$$f_{ll}^{(2)},\ f_{ll\pm 2}^{(2)},\ g_{ll}^{(l\pm 1)},\ g_{ll\pm 2}^{(l\pm 1)},\ h_{11}^{0},\ h_{11}^{(2)},\ h_{13}^{(2)} \,.$$

Thus, finally, the matrix elements (5) become single sums of expressions containing the integrals (15), (16), and (17):

$$(n'|V|n) = \frac{2\pi}{\sqrt{p'p}} \sum_{l'=0}^{\infty} A_{l'}(n';n) P_{l'm'}(\cos\theta) e^{im'\phi} \,. \tag{18}$$

In general, A_l is complex and depends even in absolute magnitude on the phases η_l of equations (11) and (13). It will be noticed that the factors $1/\sqrt{p'}$ and $1/\sqrt{p}$ which arise from equations (11) and (13), respectively, appear explicitly in equation (18), so that the expressions A_l contain p, p' only through F_l and $F'_{l'}$. Hence, for small velocities in which we are primarily interested A_l should be roughly independent of velocity. The presence of the factor $e^{im'\phi}$

[10] *Phil. Trans.*, **228**, 192, 1929.

could have been inferred from the conservation of the component of momentum about the axis $\theta = 0$ (direction of incident electrons). For the component in this direction we have initially $M + m_s$, since the incoming partial waves all have $m = 0$. For the final state, on the other hand, a partial wave with angular momentum l' has a component m', while the ion and spin contribute $M' + m_s'$. Hence, $m' = M - M' + m_s - m_s'$, the same for all partial waves.

For the integral cross-section, we require

$$\int_0^{2\pi} d\phi \int_0^{\pi} \sin\theta d\theta (n|V|n')(n'|V|n) = \frac{8\pi^3}{p'p} \sum_{l'} A_{l'}(n'; n) A_{l'}(n; n') \, .$$

The final cross-section (6) is, thus,

$$\sigma(JLSp \rightarrow J'L'S'p') = \frac{\pi}{2J + 1} \frac{1}{p^2} \sum_{l'} \sum_{MM'm_s m_s'} A_{l'}(n'; n) A_{l'}(n; n') \, . \tag{19}$$

In contrast to the expression for the individual A_l's, the summed expression is real and is independent of the phases $\eta_{l'}'$, η_l. Further, in line with what has already been said, the sum in equation (19) is only slightly dependent on p, p', provided they are small, and hence the cross-sections in the first approximation for small velocities are inversely proportional to the square of the initial velocity. This is a direct consequence of the Coulomb field, which is responsible for the form of the continuous wave functions (11) and (13). Instead of designating the partial cross-sections, as in (19), by the angular momentum l' of the outgoing electron, we adopt a more convenient and symmetrical enumeration in terms of the mean $\lambda = (l + l')/2$ of the initial and final electronic angular momenta. Then we can write

$$\left.\begin{aligned} &(2J + 1)p^2\sigma(JLSp \rightarrow J'L'S'p') \\ &\qquad = (2J' + 1)p'^2\sigma(J'L'S'p' \rightarrow JLSp) \\ &\qquad = \pi\Omega(JLSp; J'L'S'p') \\ &\qquad = \pi \sum_{\lambda} \Omega_\lambda(JLSp; J'L'S'p') \, , \end{aligned}\right\} \tag{20}$$

where the Ω_λ are symmetrical in their two sets of arguments, and the initial and final states of the system are indicated by the arrows. For

the various transitions between the states $^3P_0$, $^3P_1$, $^3P_2$, $^1D_2$, $^1S_0$, we find

$$
\left.
\begin{aligned}
\Omega_0(^3P_0p;\, ^3P_1p) &= 8(g_{11}^{(0)})^2 \\
\Omega_0(^3P_1p;\, ^3P_2p) &= 10(g_{11}^{(0)})^2 \\
\Omega_1(^3P_0p_1;\, ^1Sp_2) &= \tfrac{1}{3}\Omega_1(^3P_1p_1;\, ^1Sp_2) = \tfrac{1}{5}\Omega_1(^3P_2p_1;\, ^1Sp_2) \\
&= \tfrac{4}{3}[g_{11}^{(0)} - g_{11}^{(2)} + h_{11}^{(0)} - h_{11}^{(2)}](p_1;\, p_2) \\
&\quad \times [g_{11}^{(0)} - g_{11}^{(2)} + h_{11}^{(0)} + 2h_{11}^{(2)}](p_2;\, p_1) \\
\Omega_1(^3P_0p_1;\, ^1Dp_2) &= \tfrac{1}{3}\Omega_1(^3P_1p_1;\, ^1Dp_2) = \tfrac{1}{5}\Omega_1(^3P_2p_1;\, ^1Dp_2) \\
&= 6[g_{11}^{(0)} + h_{11}^{(0)} - h_{11}^{(2)}][g_{11}^{(0)} + h_{11}^{(0)} + \tfrac{1}{5}h_{11}^{(2)}] \\
&\quad + \tfrac{2}{3}[g_{11}^{(0)} - g_{11}^{(2)} + h_{11}^{(0)} - h_{11}^{(2)}] \\
&\quad \times [g_{11}^{(0)} - g_{11}^{(2)} + h_{11}^{(0)} + \tfrac{1}{5}h_{11}^{(2)}] + \tfrac{6}{5}g_{11}^{(2)}g_{11}^{(2)} \\
&\quad + 6g_{02}^{(1)}g_{20}^{(1)} + 6g_{20}^{(1)}g_{02}^{(1)} \\
\Omega_1(^1Dp_1;\, ^1Sp_2) &= \tfrac{4}{15}[-12f_{11}^{(2)} + 5g_{11}^{(0)} + g_{11}^{(2)} + 5h_{11}^{(0)} + h_{11}^{(2)}] \\
&\quad \times [-12f_{11}^{(2)} + 5g_{11}^{(0)} + g_{11}^{(2)} + 5h_{11}^{(0)} + 10h_{11}^{(2)}] \\
&\quad + 8[2f_{02}^{(2)} - g_{02}^{(1)}][2f_{20}^{(2)} - g_{20}^{(1)}] \\
&\quad + 8[2f_{20}^{(2)} - g_{20}^{(1)}][2f_{02}^{(2)} - g_{02}^{(1)}] \\
\Omega_1(^3P_0p;\, ^3P_1p) &= 5[g_{11}^{(0)} + \tfrac{4}{5}g_{11}^{(2)} + h_{11}^{(0)} - h_{11}^{(2)}]^2 \\
&\quad + 9(g_{11}^{(2)})^2 + 2(g_{02}^{(1)})^2 \\
\Omega_1(^3P_0p;\, ^3P_2p) &= 5[-\tfrac{8}{5}f_{11}^{(2)} + g_{11}^{(0)} + h_{11}^{(0)} - h_{11}^{(2)}]^2 \\
&\quad + [\tfrac{8}{5}f_{11}^{(2)} - g_{11}^{(2)}]^2 + \tfrac{576}{25}(f_{11}^{(2)})^2 \\
&\quad + 64(f_{02}^{(2)})^2 - 32f_{02}^{(2)}g_{02}^{(1)} + 10(g_{02}^{(1)})^2 \\
\Omega_1(^3P_1p;\, ^3P_2p) &= \tfrac{35}{2}[-\tfrac{36}{35}f_{11}^{(2)} + g_{11}^{(0)} + \tfrac{2}{7}g_{11}^{(2)} + h_{11}^{(0)} - h_{11}^{(2)}]^2 \\
&\quad + \tfrac{2376}{35}(f_{11}^{(2)})^2 + \tfrac{108}{35}f_{11}^{(2)}g_{11}^{(2)} + \tfrac{225}{14}(g_{11}^{(2)})^2 \\
&\quad + 144(f_{02}^{(2)})^2 - 72f_{02}^{(2)}g_{02}^{(1)} + 25(g_{02}^{(1)})^2
\end{aligned}
\right\}
\tag{21}
$$

EVALUATION OF THE RADIAL INTEGRALS

The integrals (15), (16), and (17) can immediately be simplified. In the first two the integration over r_2 can be performed with the help of the normalizing condition $\int_0^\infty R_2^2 dr_2 = 1$. Also, in (15) the integral over r_1 is elementary, with the form (7) for R_1. If we let

$$
I_s(r_3) = \int_0^\infty dr_1 R_1^2 \gamma_s(r_1;\, r_3)\;, \tag{22}
$$

then (15) becomes

$$f_{l'l}^{(s)}(p';\,p) = \int_0^\infty dr_3 I_s(r_3) F'_{l'}(r_3) F_l(r_3)\,, \tag{23}$$

and (16) is

$$g_{l'l}^{(s)}(p';\,p) = \int_0^\infty \int_0^\infty dr_1 dr_3 R_1 R_2 \gamma_s(r_1,\,r_3) F'_{l'}(r_1) F_l(r_3)\,. \tag{24}$$

Similarly,

$$h_{l'l}^{(s)}(p';\,p) = \int_0^\infty dr_2 R_2 F'_{l'}(r_2) \cdot \int_0^\infty dr_3 I_s(r_3) R_3 F_l(r_3)\,. \tag{25}$$

The remaining integrations have to be carried out numerically, and for this purpose tables of the Coulomb functions $F_l(r)$ are necessary. Each of the integrals is a function of the two momenta p, p', with the difference between their squares fixed by the excitation energy of the ion for the transition under consideration. To shorten the work the smaller of p, p' was taken equal to zero, and the remaining p was taken to correspond to the transition energy. Thus, for each transition each integral was computed only once and not at several points, to give the dependence on velocity. Hence, only the first approximation, in which the cross-sections vary as the inverse square of the incident velocity, was obtained. The values of the integrals computed are given in Table 1. In Table 2 are shown the corresponding numerical values of Ω computed from equations (21).

Extensive tables of the Coulomb functions $F_l(r)$ were prepared for $l = 0, 1, 2$ over the ranges of p and r appropriate for our problem. Lack of space forbids their reproduction here.

COLLISIONAL EXCITATION IN NEBULAE

In the present paper we shall confine our attention to a single illustrative problem, reserving further applications for a later communication. We now transform equation (20) from atomic to general and c.g.s. units, for which

$$\sigma_{AB} = \frac{1}{2J_A + 1}\,\frac{h^2}{4\pi m^2}\,\frac{\Omega(A,\,B)}{v^2} = \frac{4.17}{2J_A + 1}\,\frac{\Omega(A,\,B)}{v^2}\,cm^2\,. \tag{26}$$

In this equation, A symbolizes the initial and B the final atomic level. The formula may be applied to calculations of collisions of either the first or second kind.

TABLE 1

VALUES OF f, g, AND h INTEGRALS

$(p_1;\ p_2)$ $(p_2;\ p_1)$	$^3P-{}^1S$ $p_1^2-p_2^2=0.394$	$^1D-{}^1S$ $p_1^2-p_2^2=0.209$	$^3P-{}^1D$ $p_1^2-p_2^2=0.185$	$^3P-{}^3P$ $p_1^2-p_2^2\cong 0$
$f_{11}^{(2)}$	....	+0.098	....	+0.126
$f_{02}^{(2)}$	....	− .045 − .050	....	− .045
$f_{20}^{(2)}$	....	− .050 − .045	....	− .045
$g_{00}^{(1)}$	....	....	....	+ .072
$g_{11}^{(0)}$	−0.203 − .297	− .240 − .293	−0.246 − .292	− .288
$g_{11}^{(2)}$	+ .098	+ .103	+ .103	+ .104
$g_{02}^{(1)}$	....	− .046 − .048	− .046 − .048	− .047
$g_{20}^{(1)}$	....	− .048 − .046	− .048 − .046	− .047
$h_{11}^{(0)}$	− .230 − .268	− .246 − .265	− .248 − .265	− .262
$h_{11}^{(2)}$	+ .084 +0.092	+ .090 +0.094	+ .090 +0.094	+0.096

TABLE 2

TARGET AREAS, Ω, FOR COLLISIONAL EXCITATION OF O III BY ELECTRON IMPACT

	$^3P_0$	$^3P_1$	$^3P_2$	^1D
^1S	0.39	1.18	1.96	2.71
^1D	2.22	6.66	11.1	
$^3P_2$	4.06	11.2		
$^3P_1$	1.72			

For electrons with initial velocities corresponding to the transition energy, the σ's are as follows:

$$\left.\begin{aligned} \sigma(^3P_0\rightarrow{}^3P_1) &= 1.69\times 10^3\sigma_H\,, & \sigma(^3P_0\rightarrow{}^3P_2) &= 1.45\times 10^3\sigma_H\,,\\ \sigma(^3P_0\rightarrow{}^1D_2) &= 12.0\sigma_H\,, & \sigma(^3P_0\rightarrow{}^1S_0) &= 9.91\sigma_H\,, \end{aligned}\right\} \quad (27)$$

where σ_H is the theoretical area, πa^2, of the 1s orbit of hydrogen,

$$\sigma_H = 8.765 \times 10^{-17}\ \text{cm}^2\ . \tag{28}$$

The target areas are large, especially for excitation within the levels of ^3P.

We assume that the velocity distribution of the nebular electrons is Maxwellian, defined by a kinetic temperature T_ϵ, so that the fraction $f(v)$ of electrons having velocities between v and $v + dv$ is

$$f(v)dv = 4\pi\left(\frac{m}{2\pi kT_\epsilon}\right)^{3/2} v^2 e^{-\frac{1}{2}mv^2/kT_\epsilon}dv\ . \tag{29}$$

Then the total number of collisional excitations (or de-excitations) from level A to B, per cubic centimeter per second,[11] is

$$\mathfrak{F}_{AB} = N_A N_\epsilon \int_{v_0}^{\infty} \sigma_{AB} v f(v) dv\ , \tag{30}$$

where v_0 is the lowest velocity capable of inducing the transition; v_0 is zero for superelastic collisions and is defined by

$$\tfrac{1}{2}mv_0^2 = \chi_{AB} \tag{31}$$

for inelastic collisions. χ_{AB} is the energy difference between the levels A and B. Carrying out the integration, we find for inelastic impacts,

$$\left.\begin{aligned}\mathfrak{F}_{AB} &= N_A N_\epsilon \frac{\Omega(AB)}{\pi_A}\frac{h^2}{2\pi m^2}\left(\frac{m}{2\pi kT_\epsilon}\right)^{1/2} e^{-\chi_{AB}/kT_\epsilon} \\ &= 8.54 \times 10^{-6}\frac{N_A N_\epsilon}{T_\epsilon^{1/2}}\frac{\Omega(AB)}{\tilde{\omega}_A} e^{-\chi_{AB}/kT_\epsilon}\ .\end{aligned}\right\} \tag{32}$$

For superelastic impacts, the equation is

$$\mathfrak{F}_{AB} = 8.54 \times 10^{-6}\frac{N_B N_\epsilon}{T_\epsilon^{1/2}}\frac{\Omega(AB)}{\tilde{\omega}_B}\ . \tag{33}$$

[11] R. H. Fowler, *Statistical Mechanics*.

We have written $\bar{\omega}_A$ for $2J_A + 1$, the statistical weight, in the above equations.

When the gaseous assembly permits detailed balancing, so that

$$\mathfrak{F}_{AB} = \mathfrak{F}_{BA} , \tag{34}$$

we have the condition

$$\frac{N_B}{N_A} = \frac{\pi_B}{\bar{\omega}_A} e^{-\chi_{AB}/kT_\epsilon} , \tag{35}$$

which is the Boltzmann relationship for thermodynamic equilibrium. If the only means of removing atoms from levels B and A are collisional, equation (35) will hold exactly, even for gaseous nebulae. Leakage of atoms from level B by spontaneous emissions will cause departure from the Boltzmann formula. It is easy to show, however, that (35) will still be a fair approximation in any nebula where the probability of de-excitation of B by superelastic impact exceeds that for spontaneous transition, i.e., when

$$\sum_A \mathfrak{F}_{BA} > N_B \sum_A A_{BA} , \tag{36}$$

where A_{BA} is the Einstein coefficient of spontaneous emission.

Of particular interest is the relative population in the levels of ^{3}P. In Paper IX of this series Shortley and Menzel discussed the fractional multiplets observed in gaseous nebulae. The spectra suggest an initial excitation of the single level 2p3d $^3P^0_2$, with a subsequent cascade to the levels of the lower configurations 2p3p and 2p3s. The source of excitation, as Bowen has pointed out, is probably the ultimate line λ 303 of *He* II, which must be very intense in certain nebulae. The line 2p^2 $^3P_2$ — 2p3d $^3P^0_2$ of *O* III coincides very closely with this helium line, the resonance absorption of which by *O* III will lead to selective excitation of $^3P^0_2$, as observed.

For the Bowen fluorescent process to work effectively, however, requires that the population of *O* III atoms in the level 2p^2 $^3P_2$ be large. Since $^3P_2$ has a mean lifetime of 2.9 hours,[12] there will be an

[12] *Ap. J.*, **91**, 307, 1940.

effective population in that level only if the inequality (36) is fulfilled.

$$\sum A_{BA} = \frac{1}{2.9} \times 3600 \sim 10^{-4} . \tag{37}$$

$$\sum \mathfrak{F}_{BA} = 8.54 \times 10^{-6} \frac{N_B N_\epsilon}{10^2} \frac{\Omega(^3P_2, {}^3P_1) + \Omega(^3P_2, {}^3P_0)}{5} , \tag{38}$$

for $T_\epsilon = 10^4$ deg. Hence, from (36) we obtain the condition that

$$N_\epsilon > 4 \times 10^2 . \tag{39}$$

An independent investigation, the details of which will be given in a later paper, shows that the brighter nebulae, in particular the ones showing the lines of the fluorescent mechanism, have electron densities somewhat in excess of this figure. We conclude, therefore, that collisions occur with sufficient frequency to maintain approximately a Boltzmann distribution among the levels of ^3P.

Dunham observed that interstellar lines of *Ti* II appear from only the lowest atomic level. The excitation parameters for *O* III are, of course, not directly applicable to *Ti* II. But the figures, which indicate a high accumulation of atoms in the lowest level as a result of the very low electron density in interstellar space, are consistent with the observational data.

Extension of the calculations to the $^1D_2$ and $^1S_0$ levels of *O* III leads to an independent determination of the electron temperature and of the abundance of *O* III. These results will be given in a later paper.

We wish to acknowledge the advice and assistance of Dr. J. H. Van Vleck. A grant to Dr. Van Vleck from the Milton Fund and Clark Bequest of Harvard University enabled Mr. Hebb to pursue the present investigation.

DUKE UNIVERSITY
AND
HARVARD UNIVERSITY
April 1940

RELATIVE FREQUENCIES OF METEORS

J. D. WILLIAMS

ABSTRACT

On the assumption that the number of meteors increases logarithmically with their magnitudes, the base κ is evaluated from a series of binocular observations by the method of maximum likelihood. The reliability of the value so found, 2.08, is discussed, following which an estimate is made of the space density of meteoric material.

INTRODUCTION

The writer published[1] recently a small quantity of data from which an inference can be made on the relative numbers of meteors in successive magnitude classes and, tentatively, on the real number. Material even approximately suited to the question is difficult to find which may justify basing an estimate on so few observations.

A reasonable empirical approach is to assume a simple exponential law relating the number of meteors to their magnitudes, as was done by Hoffmeister,[2] by Watson,[3] and by the writer.[4] It cannot be said that Hoffmeister verified the exponential character of the relation, since his observations seemed to indicate a more rapid increase, which, together with his physical theory of meteors, would have led to infinite mass for meteoric material. Watson, using a telescopic series by Boothroyd and naked-eye observations by Öpik, obtained a constant ratio of increase in numbers per magnitude equal to 4. As a check, he discussed Hoffmeister's data on naked-eye meteors, a group of telescopic observations reported by Öpik, and observations of his own, which led, respectively, to $\kappa = 3.5$, 3.2, and less than 4, though a re-examination of his telescopic observations[5] gave the value 2.7. The writer examined the question, using naked-eye and binocular observations, and established that κ had the same value at two points on the magnitude scale some 5 or 6 mag. apart; this was done by choosing for comparison two short magnitude intervals

[1] *Proc. Amer. Phil. Soc.*, **81**, No. 4, 505, 1939.

[2] *Veröff. Berlin-Bablesberg*, **9**, No. 1, 1, 1931.

[3] *Harvard Ann.*, **105**, No. 32, 623.

[4] *A.J.*, No. 1110, 1939.

[5] *Proc. Amer. Phil. Soc.*, **81**, No. 4, 493, 1939.

where the inevitable processes of selection could be expected to be the same. The values of κ at the two points were in excellent agreement, thus strongly supporting the hypothesis; but the actual value of $\kappa = 2.29 \pm 0.16$ had small weight, since preferential selection, due to half a dozen causes, might be playing its part. In the present paper we seek to remedy this by choosing from the binocular series those observations best suited to a determination of κ, *assuming* its constancy.

THE DATA

The observations were made by Mr. R. Knabe, under conditions already described by me.[6] The frequencies of initial points f_i appearing in Table 1, herewith, were selected from the list in the publication cited[7] to fit the following conditions: the initial point was observed and was, at most, 3° from the center of the field, the observer's eyes being directed to the latter; when more than one estimate of magnitude was given, that corresponding to maximum light is adopted here—for several reasons which need not be enumerated, beyond remarking that this may be most intimately related to the mass of the particle. We thus have a group of objects giving, presumably, a representative time average of the observable flux of meteoric material, chiefly of the sporadic variety, since shower dates are excluded.

TABLE 1

r \ m	1	2	3	4	5	6	7	8	9
0°.5	0	0	1	4	3	2	8	8	4
1.0	0	1	0	2	6	4	15	28	5
1.5	0	3	0	1	4	6	25	38	7
2.0	1	1	0	4	6	9	28	46	2
2.5	0	0	0	3	6	11	22	36	5
3.0	2	1	0	3	4	12	24	27	2
f	3	6	1	17	29	44	122	183	25

Area	F	κ	σ_κ
1	1.00	1.75	$\pm$0.23
3	0.52	2.04	$\pm$ .28
5	0.43	2.22	$\pm$ .29
7	0.39	2.11	$\pm$ .23
9	0.26	2.31	$\pm$ .31
11	0.23	2.00	$\pm$0.21

The data in Table 1 are given as functions of apparent magnitude m and angular distance r from the line of sight, class intervals being

[6] *Proc. Amer. Phil. Soc.*, **81**, No. 4, 505, 1939.

[7] *Ibid.*, Table 2. If the jth digit in column C is designated $C(j)$, then specifically the selection was such that $C(2) = 1, 2, \ldots, 6$; $C(3) = 1, 4$; $C(4) \neq 1, 2$.

typified by $0^m.75 < m = 1 < 1^m.75$ and $0^\circ.5 < r = 1 < 1^\circ.0$. (Magnitude estimates were in $0^m.5$ intervals but are here combined to augment the frequencies.) The size of the first zone, $r = 0^\circ.5$, is such that it was observed almost exactly by the macula lutea; and in terms of this zone as a unit the areas of the successive zones increase as do the odd integers. It is immediately evident that the observed frequencies in the columns do not, in general, increase in this manner. When the frequencies, say those brighter than 8^m, are reduced to equal areas, they are found to decrease very nearly linearly as one recedes from the foveal region, excluding the central zone itself, where the frequencies are even higher; this may be seen from the figures in column F of Table 1. Since the stellar threshold for some regions of the eye was fainter than 10^m and, further, since a large number of the faint objects seen lie in these outer regions, it is difficult to ascribe the area failure to lack of sensitivity. More probably it means that the observer is very attentive to events in the foveal region and progressively less so to events that occur off the line of sight. If this interpretation is correct, there is no reason why (in, say, the fourth zone) a larger percentage of 6^m objects should be missed than of 4^m objects, since both classes are far above the threshold. It can be established easily that loss of a fixed fraction of the total in each magnitude class will not affect the value of κ deduced by the method used here.

It will be observed that the run of the frequencies is very rough because of errors in classification (principally with regard to m) and to the small size of the sample. We propose to ignore the roughness and to obtain the most probable value of κ for each zone by an objective method, namely, that of maximum likelihood.[8]

THE ANALYSIS

Suppose that the magnitude range is cut into $s + 1$ intervals by the points

$$-\infty,\ m_0,\ m_0 + \Delta m,\ m_0 + 2\Delta m,\ \ldots\ ,\ m_0 + s\Delta m = m' .$$

[8] R. A. Fisher, *Proc. Cambridge Phil. Soc.*, **22**, 700, 1925.

Then the probability p_i of finding in our sample a meteor whose magnitude lies in the $(i + 1)$th interval is

$$p_i = A \int_{m_0+(i-1)\Delta m}^{m_0+i\Delta m} e^{am} dm \qquad (i = 1, 2, \ldots, s)$$

and

$$p_0 = A \int_{-\infty}^{m_0} e^{am} dm \qquad (e^a = \kappa),$$

where A is fixed by the normalization

$$1 = A \int_{-\infty}^{m'} e^{am} dm .$$

In order to determine a in terms of the observed frequencies $f_i (i = 0, 1, \ldots, s)$, it can be established directly that the *likelihood* of the sample,

$$P = \prod_0^s p_i^{f_i},$$

attains its maximum when

$$\coth \tfrac{1}{2} a\Delta m = 1 + \frac{2}{N - f_0}\left[Ns - \sum_1^s i f_i\right],$$

where N is the total observed frequency. This results when we equate to zero the logarithmic derivative of P with respect to a. The equation gives an optimum estimate of the parameter a. Its standard deviation may be estimated from

$$\sigma_a = \left[E\left(-\frac{d^2 \log P}{da^2}\right)\right]^{-1/2} = \frac{2 \sinh \tfrac{1}{2} a\Delta m}{\Delta m \sqrt{N(1 - p_0)}},$$

where E indicates the operation of taking the mean value.[9] We pass to the standard deviation of κ by means of $\sigma_\kappa = \kappa\sigma_a$.

[9] *Ibid.*

From these equations and from the data in Table 1 we compute κ and σ_κ for each zone, using only meteors brighter than $7^{m}.75$, so as to stay well away from the threshold. The values, which appear under κ in the same table, are scattered, as must be expected from rough samples, and there is no certain systematic variation with respect to r. Combining the zones leads to

$$\kappa = 2.08 \pm 0.10 \, .$$

Table 2 shows the computed frequencies.

TABLE 2

m	1.0	1.5	2.0	2.5	3.0	3.5	4.0	4.5	5.0	5.5	6.0	6.5	7.0	7.5	8.0	8.5	9.0	9.5
Observed	1	2	0	6	0	1	10	7	9	20	21	23	51	71	105	78	22	3
Computed	0.6	0.8	1.2	1.7	2.5	3.6	5.2	7.6	10.9	15.7	22.7	32.7	47.2	68.0	98.3	142	205	296

SOURCES OF ERROR

This value of κ differs so considerably from those determined previously that it becomes important to consider the errors to which it is subject. Four potential sources of error occur to the writer, and they will be discussed rather qualitatively.

a) There is the question regarding the effect of mistakes in classification, such as is suggested by the 6, 1, 17 distribution from 2^{m} to 4^{m} in Table 1. It happens there was a centrally located comparison star of $6^{m}.7$ to which the scale was fixed; hence the 6^{m} and 7^{m} classifications may be of higher reliability than the brighter ones. If we disregard the *distribution* of those classified as 1^{m}, 2^{m}, ... , 5^{m} but retain their number as f_0, we arrive again at $\kappa = 2.08$, showing that the determination is not very dependent upon mistakes of classification in the interval where they are most likely to occur. Also, if κ is computed from the data as originally grouped (cf. Table 2), the same value is obtained. We conclude that mistakes in classification probably are not of first importance in this series.

b) Quantitatively the effects of motion on magnitude estimates are not very precisely known. Öpik states[10] that a magnification of less than 10 probably produces a negligible effect, the statement fol-

[10] *Harvard Circ.*, No. 355, p. 10.

lowing a rough experiment which led him to adopt a set of magnitude decrements 0.1, 0.2, 0.3, 0.6, etc., as corrections corresponding to angular velocities of 100, 160, 250, and 400 deg/sec. While Knabe's velocities are not known, there is reason to expect them to lie principally in the range 20–50 deg/sec on the sky or 140–350 deg/sec subjective. Part of the small error, which undoubtedly exists, will appear as a zero-point error, while only a differential effect will disturb the scale. If the true scale was thus affected by a pure expansion, amounting, for example, to as much as $0^m_.5$ in a range of 7^m, our κ would need to be increased by about 0.12. This does not promise to be a major effect.

c) There is a possibility of Purkinje effect, but this would tend to produce an artificially large value of κ. This would oppose the error due to (*b*); but it must surely be of second order, since, as shown in (*a*), our determination can be based essentially on meteors in a narrow magnitude interval so far above the threshold that we should not selectively record red meteors.

d) The circumstance that the number of initial points is discussed in conjunction with maximum light must introduce a bias because of the restricted field of view, for some meteors will attain their maximum light after an appreciable flight. The writer has a distinct impression, looking back on about 18,000 observations, that bright meteors frequently have greatest light near their end-points and, further, that the phenomenon of increasing light is much more common than is the converse. The latter is in complete agreement with Knabe's explicit estimates of changes in brightness, and his records show that it holds for faint meteors as well. If the path length, ΔL, between the initial point and that point at which maximum light occurs, decreases with increasing magnitude, as seems probable, the proportion of meteors improperly classified as to magnitude will decrease with increasing magnitude. Since the error will always be positive, κ will be increased. It is difficult to judge the size of the effect from the available data; the value 2.08 may be spuriously large or may be almost unaffected. No zone should be less affected than the central one, since there *all* meteors which attain their maximum light in less than 3° will be properly classified. It is disturbing to note that the smallest κ found, 1.75, corresponds to this zone. A set

of increments ΔL could probably be devised to account for most of the variation in κ from zone to zone, but the standard deviations do not encourage the attempt.

No other sources of error which show promise occur to the writer, so we are unable to justify the discrepancy between this determination and that of others on the basis of errors in the present data. Only one *ad hoc* hypothesis has been introduced consciously: we have assumed that for objects more than $2^{m}.75$ above the threshold, at any given point in the field of vision not more than 20° (subjective) from the line of sight, the fractional number of meteors of a given brightness that are missed does not depend (at least effectively) upon the magnitude. There seems to be confirmation of this in Öpik's[11] work on telescopic meteors where he deduces "true" frequencies from a consideration of the number of common objects seen by two observers. Comparison of his true and observed frequencies shows that the following proportions of the true numbers were missed, starting with the faintest magnitude group: 0.92, 0.47, 0.22, 0.24, and 0.18. These figures suit the hypothesis very well.

While the hypothesis seems necessary in order to reach an explicit determination, the effect (*d*) places it strongly under suspicion. But the latter may serve a useful purpose in bringing the various investigations to a common result. The usual discussion of the effective field of a telescope for meteors of given length leads to the formula

$$\pi r^2 + 2rL(m) ,$$

which is taken as a measure of the observed number of objects of length $L(m)$. This is found from considerations of geometrical probability which involve implicitly the assumption that the meteors are uniformly luminous. By means of it large factors are introduced to reduce the numbers of meteors in all magnitude classes to equal areas. But if the assumption of uniform luminosity is not fulfilled, use of the formula may cause an overcorrection and lead to too large a κ. This would happen, for example, if the meteors were uniformly luminous except for a flare at the end, "true" magnitude being based on maximum light.

[11] *Tartu Pub.*, **27**, No. 2, 5, 1930.

Watson's $\kappa = 2.7$ was deduced from data corrected by this formula. When κ is estimated from his final data by the method of maximum likelihood, it is found to be 2.64. If the faintest useful magnitude class is excluded, we find $\kappa = 2.45$; and if the next class is also omitted, $\kappa = 2.39$. These last two values, ~ 2.4, are deduced from a magnitude range comparable to that used in the present paper. The difference in the two determinations of κ, about 0.3, may well be concealed in the last formula.

SPACE DENSITY

The apparent magnitude scale used to this point may be reduced, in an approximate manner, to a zenithal magnitude scale m_z by a change in zero point of 5 log sec $57\overset{\circ}{.}4 = 1\overset{m}{.}34$. The faintest objects in the series are about 8^{m_z}, which, if the maximum light occurs at about 85 km, corresponds to an absolute magnitude of 71^M.

These data are very limited for an attack on the problem of absolute numbers. One can do no better than to accept the rate established in the best observed zone, the central one. Here the number of objects is 18. We can get some idea of its stability by appealing to the Poisson law for events of small probability, which leads to the statements that the probability is 0.44 that a repetition of the series would yield a number in the range 18 ± 3, 0.59 that it would lie within 18 ± 4, and 0.96 that it would lie within 18 ± 9.[12]

The average area under observation at zenith distance z is approximately

$$\pi r^2(h^2 + \sigma_h^2) \sec^3 z ,$$

where h is the initial height and σ_h^2 its variance. The latter is of the order of 100 when the kilometer is the unit and we adopt the value $h = 95$ km, which leads to an area of 14.0 km² for the central zone. The initial height is a very satisfactory quantity statistically; Öpik's study of 3,500 heights shows it to be largely free from systematic changes such as affect terminal heights.[13] The actual observing time was 163.6 hours, allowance being made for the plotting time.

[12] K. Pearson, *Tables for Statisticians and Biometricians*, **1**, 1930.

[13] *Harvard Ann.*, **105**, No. 30.

From these data we find the flux of particles brighter than $m_z = 6.0$ to be 1.62×10^{-16} cm^{-2} sec^{-1}, which implies about 73,000,000 megascopic objects daily for the whole earth. This may be compared to Watson's value of about 127,000,000 over the same range. The particle flux n due to those whose magnitudes lie between m_z and $m_z + dm_z$ is

$$n = 1.46 \times 10^{-18+0.319m_z} dm_z \text{ cm}^{-2} \text{ sec}^{-1} .$$

Using a result from kinetic theory[14] which connects n and the average velocity C (say 64 km/sec), namely, $4n = \nu C$, we obtain for the partial molecular or particle density ν the relation

$$\nu(m_z)dm_z = 9.1 \times 10^{-25+0.319m_z} dm_z \text{ cm}^{-3} .$$

We find, for example, about 60,000 meteors] 6^{m_z} in a space equal to that occupied by the earth.

Watson has used a formula of Öpik relating velocity and magnitude to mass, which was such as to make the mass of a second-magnitude Perseid 12 mg. In our case it reduces to

$$0.054 \times 10^{-0.4m_z} \text{ gm} ,$$

when the material density is 5, and leads to a partial space density ρ amounting to

$$\rho(m_z)dm_z = 4.9 \times 10^{-26-0.081m_z} dm_z \text{ gm cm}^{-3} .$$

We may estimate only the total space density associated with faint meteors, since the density increases without limit if we sum over large negative values of m_z. Thus,

$$\rho(m[m_z) = 2.6 \times 10^{-25-0.081m_z} \text{ gm cm}^{-3} .$$

The density corresponding to particles of mass less than a gram ($m_z = -3.2$) is 4.9×10^{-25} gm cm^{-3}; all but 12 per cent of this is due to meteors] 8^{m_z}.

The value $\kappa = 2.08$ found here leads to finite space densities ac-

[14] E.g., J. Jeans, *Dynamical Theory of Gases*, p. 323, 1928.

cording to the physical theories of Hoffmeister and of Öpik, when the summation is extended to all possible high-magnitude numbers, being less than the permitted maxima, ~ 2.8 and ~ 2.5, respectively. But the whole idea of a simple exponential law is evidently a fiction induced by the small observational range. The data would satisfy other laws which may have more significance; for instance, by making the exponent quadratic rather than linear in m, we would have the usual Gaussian function, which would give finite mass regardless of the region of integration. If the two disposable parameters were chosen so as to make the mean fall near 16^m and the dispersion equal to about 4^m, the observations would be very satisfactorily represented. But the law assumed in this paper cannot be rejected on the basis of the present observations, for, when a χ^2-test for goodness of fit[15] is applied to the data in Table 2, we find that the probability is 0.5 that a less favorable set of frequencies will result from sampling fluctuations.

PRINCETON UNIVERSITY OBSERVATORY
April 7, 1940

[15] Pearson, *op. cit.*

PROPER MOTIONS IN THE GALACTIC CLUSTER MESSIER 39

E. G. EBBIGHAUSEN

ABSTRACT

Proper motions are derived from Yerkes plates for an area of about 30′ diameter centered on the cluster M 39. The resulting motions are contained in Table 2. Plots of the motions are found in Figs. 1, *A* and *B*. The spectral diagram of the cluster is given in Fig. 2.

This paper gives the results of the determination of proper motions in the galactic cluster Messier 39 (NGC 7092; $\alpha = 21^h28^m6$, $\delta = 48°00'$; $l = 60°5$, $b = 2°7$). For this cluster Trumpler gives a distance of 330 parsecs, an angular diameter of 32′, and the type is designated as "1a," indicating that the main sequence runs up to spectral type A but that there are no late-type giants. Assuming a distance of 330 parsecs, the distance modulus is 7.6.

TABLE 1

Old Plate	New Plate	Interval in Years	Limiting Photovisual Magnitude
2656..........	F603	23.0	11.2
2779..........	F604	22.8	11.2

Previous work on this cluster has been done by Trumpler,[1] who determined magnitudes, spectral types, and radial velocities for a number of stars. Quite a few of these stars have radial velocities clustering around −14 km/sec, which might be considered to be the radial velocity of the cluster.

The material consists of two pairs of plates taken with the 40-inch refractor of the Yerkes Observatory. Each pair of plates has two images of each star. Table 1 is a list of these plates.

The procedure adopted in this paper is the same as that used in the measurement and reduction of NGC 2548, except that in this

[1] *Pub. A.S.P.*, **40**, 265, 1928.

instance the magnitude equation receives a different treatment. Of the total of 50 stars whose motions were measured, 34 were measured in all four sets. The individual motions in both co-ordinates were freed of zero-point differences and averaged to form a mean. These mean motions were then plotted. Near the origin there appeared a fairly loose grouping of stars, from which 24 were selected as provisional cluster members. The individual motions of this group from each set of measures were then plotted against the photovisual magnitudes. These magnitudes were rough determinations made with the 10-inch Bruce photographic refractor of the Yerkes Observatory, using a I-G plate and a yellow filter with the North Polar Sequence as standard. These plots of motion against magnitude showed such a large scatter that it was considered impossible to determine the magnitude equation and that the provisional cluster members must contain a large percentage of field stars, thereby vitiating the method. It was therefore decided that the published motions should simply be the average of the four sets of measures with zero-point differences removed.

From an intercomparison of the four sets of measures, a mean error of about $\pm 0\overset{''}{.}0013$ per year in either co-ordinate was derived for a star which had been measured on all four sets. Because of the difficulties described herein this value of the mean error is necessarily rough. Since no significant difference in weight was found between each of the four sets, unit weight was assumed for each. The combined measures are given in Table 2.

The columns give, respectively:

1. The author's catalogue number (no previous catalogue exists).

2 and 3. The co-ordinates on a system in which $+x$ and $+y$ indicate increasing right ascension and declination, respectively. One unit = $22\overset{''}{.}6$.

4. The photographic magnitude on Trumpler's system. Most of the magnitudes were kindly supplied by Dr. Trumpler in advance of publication. For the remainder, interpolations were made on an Eastman 40 plate taken with the 24-inch reflector of the Yerkes Observatory.

5. The spectral type kindly supplied by Dr. Trumpler. A dash in

TABLE 2

No.	x	y	m_{pg}	Spec.	μ_x	μ_y	n	Class
1	31	160	7.3	A0	−24	−45	4	1
2	31	86	10.6	....	−36	+15	4	2
3	44	192	9.1	A1n	+7	+37	4	1
3a	71	226	11.2	A3:	+63	+334	2	3
4	55	180	9.2	A3n	+6	+8	4	1
5	91	172	7.8	A1n	+11	−6	4	1
6	64	148	11.5	F0:	−16	0	4	1
7	90	142	13.0	....	−14	+79	2	3
8	89	140	10.0	A3	−37	−46	4	2
9	59	137	9.1	G7	+131	−1242	4	3
10	61	129	10.9	A3	−17	−58	4	2
11	87	83	11.2	F3	−17	−112	4	3
12	83	68	12.1	....	+32	−2	4	2
13	46	63	10.6	....	+23	+85	4	3
14	46	60	11.5	....	+113	+15	4	3
15	72	53	12.8	....	+28	−22	2	2
16	97	203	10.3	A4	+44	+52	4	3
17	98	199	7.9	A1n	+3	+25	4	1
18	123	200	12.1	....	−1	+252	4	3
19	129	180	9.0	A2	+1	−2	4	1
20	120	178	9.5	A3n	−21	+15	4	1
21	132	134	11.2	F8	+93	+166	4	3
22	116	131	8.9	A3s	−35	−37	4	1
23	112	130	7.6	A1	−3	−55	4	1
24	130	102	8.0	A0n	+164	+129	4	3
25	105	102	12.0	A0	+102	+99	2	3
26	110	96	6.8	A0s	−37	−100	4	2
27	101	77	9.9	A5	+180	+244	4	3
28	105	63	11.3	F2	+2	+16	3	2
29	105	61	12.9	....	−9	−23	3	2
30	133	51	8.4	A1	−85	−186	2	3
31	143	178	8.6	A2n	−14	−7	4	1
32	146	151	11.8	M3	+38	+223	4	3
33	142	143	6.6	A0	−2	−52	4	1
34	137	129	9.6	A2n	−37	−30	4	2
35	166	130	9.0	A2	−23	−65	4	1
36	156	113	11.4	G0	+293	+24	4	3
37	147	80	12.2	K0:	+9	+68	4	3
38	140	78	8.2	A0	−42	−125	4	2
39	171	62	9.8	K0	−280	−240	4	3
40	170	56	10.7	A1	+17	−36	4	2
40a	191	192	6.8	A0s	−24	+57	2	1
41	209	44	10.7	....	+83	+111	2	3
42	191	32	10.1	....	−6	+9	2	2
43	221	129	11.8	....	−19	+93	2	3
44	216	115	11.1	....	−46	+108	2	3
45	233	107	9.1	....	−44	−125	2	2
46	239	81	9.1	:.......	−25	−15	2	1
47	230	67	10.1	:.......	+46	−16	2	2
48	235	57	10.7	....	−9	+11	2	2

place of the spectral type indicates that Trumpler supplied the magnitude but gave no spectral type.

6 and 7. The proper motions in x and y, respectively, expressed in units of 0".0001 per year.

8. The number of times the star was measured.

9. A measure of the probability that the star is a cluster member.

Although the process, described above, of combining the results should, on the average, reduce the magnitude equation, an indeterminate amount remains, and hence the separation of cluster

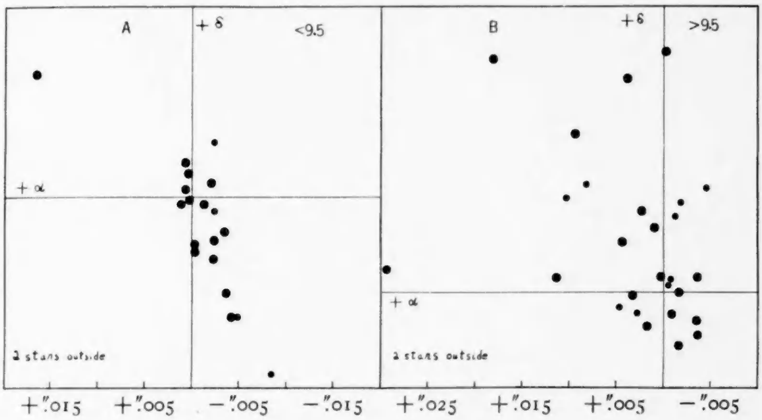

FIG. 1.—Large dots indicate stars measured four times; small dots, twice

members from the field stars will not be as complete as it might otherwise be. A visual inspection of the cluster and the surrounding region indicates that the number of faint cluster members is very small. Figures 1, *A* and *B*, are proper-motion plots for two groups of stars, respectively—those brighter and those fainter than photovisual magnitude 9.5. The 14 stars in Figure 1, *A*, near the origin are considered to be cluster stars, and the remainder to be less probable members.

The number of stars brighter than 9.5 mag. and between 9.5 and 11.0 mag., respectively, were counted in the region of the cluster and in a large area surrounding it. It is found that in the cluster 20 stars are brighter than 9.5, whereas the expected number of field stars is 4.5 ± 2. Hence, 16 ± 2 stars must be cluster members, and 4 ± 2 field

stars. Inspection of Figure 1, *A*, shows a concentration of 14 stars near the origin, and some outlying stars. One is justified, therefore, in identifying the former group as cluster stars, and the designation "class 1" is assigned to them. The three stars just outside this group may, in part, be cluster members, and the designation "class 2" is assigned to them. The remaining 4 stars, of which one is outside

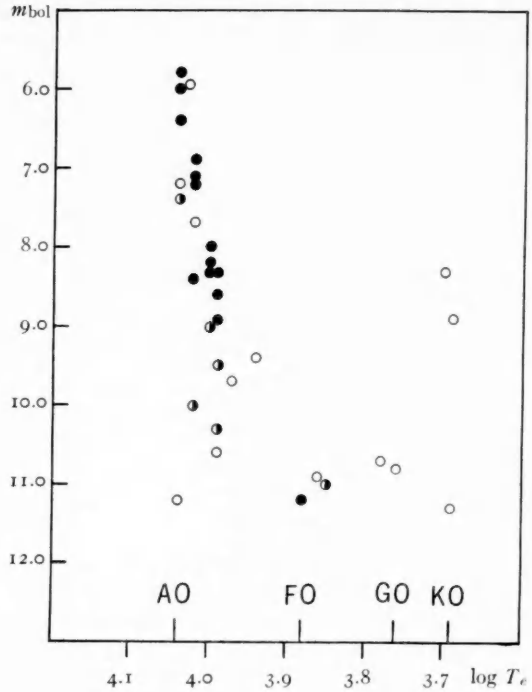

FIG. 2

the diagram, are very probably field stars and are designated as "class 3."

For the stars between photovisual magnitude 9.5 and 11.0 the counts give 32 stars within the cluster area. However, the expected number of field stars is 31 ± 5.5, so that from the counts alone there is no positive evidence of cluster members fainter than 9.5. Therefore, Figure 1, *B*, must consist largely of field stars, even the inner region. "Class 2" is assigned to the stars of the inner region; and "class 3," to the remainder, which are very probably field stars.

Figure 2 is the spectrum-magnitude diagram of the cluster. The abscissas are given both as log T_e and spectral type. The ordinates are bolometric magnitudes. The temperature scale is the same as that used in Kuiper's[2] paper. Class 1 is represented by filled circles; class 2 by half-filled circles; and class 3 by open circles.

Four stars of class 1 are found in the *Boss General Catalogue*. Their weighted mean annual proper motion is $\mu_a = -0\overset{''}{.}016 \pm 0\overset{''}{.}003$ and $\mu_\delta = -0\overset{''}{.}018 \pm 0\overset{''}{.}003$, or $0\overset{''}{.}024 \pm 0\overset{''}{.}003$ in position angle 222°. If all four stars are cluster members, the above value might then be considered as the absolute proper motion of the cluster.

Grateful acknowledgment is made to Dr. R. J. Trumpler for making his data available in advance of publication. I wish to express my thanks to Dr. Struve for having placed the facilities of the Yerkes Observatory at my disposal and to thank Dr. Kuiper for the help that he has given me in the discussion.

YERKES OBSERVATORY
June 1940

[2] *Ap. J.*, **86**, 176, 1937.

NOTE ON ALGOL

Dr. Philip Taylor, of the Flower Observatory, has very kindly called my attention to an error in my paper concerning Algol (*Ap. J.*, **90**, 479, 1939). In deriving the bolometric difference in magnitude between the eclipsing components from the visual observations, I applied the color correction in the wrong sense. The bolometric difference should read $3^{m}_{\cdot}2$ instead of $1^{m}_{\cdot}8$. Since an assumption was made to increase this erroneous value of $1^{m}_{\cdot}8$ to $3^{m}_{\cdot}3$, it is evident that this assumption, for purposes of discussion, balanced the error. Consequently, the elements suggested on pages 482 and 483 follow directly from the data, without the assumption.

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ERRATA

The following corrections should be made in the article by J. H. Oort, "Some Problems concerning the Structure and Dynamics of the Galactic System and the Elliptical Nebulae NGC 3115 and 4494," which appeared in the April, 1940, issue of Volume **91**:

P. 278, line 18. *For* distance *read* distant.

P. 282, line 9. The formula *should read* $\Theta - \Theta_{cS} = 2Bx$.

P. 282, line 11. The formula *should read* $\Pi = an \cos nt = -2By$.

P. 282, second line from bottom. Delete the minus sign in the last term.

P. 287, Fig. 3. The numbers giving the scale of the ordinates should be diminished by 0.12.

P. 290, Fig. 4. Just below the figure, *for* log Z'' *read* log z''. *Insert:* "The values of the ordinates for the dashed curve (log φ) are one unit smaller than the numbers indicated for log I."

P. 291, Fig. 5. For Z'' read z''. The ordinate number 10 should be lifted so as to correspond with the second horizontal line.

P. 295, Fig. 7. For Z'' read z''.

P. 300, first line below Table 3. *For* Θ_2 *read* θ_2.

P. 302, Table 4, fourth column. *For* $18.6/\kappa^3$, $6.02/\kappa^3$, and $0.56/\kappa^3$ *read* $18.6/\kappa$, $6.02/\kappa$, and $0.56/\kappa$.

P. 302, Table 4, last column. *For* 19.6κ, 23.9κ, and 249.0κ *read* $19.6/\kappa$, $23.9/\kappa$, and $249.0/\kappa$.

THE DYNAMICS OF STELLAR SYSTEMS. IX–XIV*

S. CHANDRASEKHAR

ABSTRACT

The kinematical characteristics which are postulated in this paper for describing the state of motions in a stellar system are the same as in the earlier paper (see Abstract of Parts I–VIII) except for the assumption of the steady state, which is now dropped. As in Parts I–VIII, the mathematical problem reduces to finding the circumstances under which the equation of continuity regarded as a partial differential equation for the *distribution function* Ψ in the seven variables x, y, z, U, V, W, and t admits of a solution of the form

$$\Psi \equiv \Psi(Q + \sigma[x, y, z, t]) , \qquad \text{(i)}$$

where Q is a general homogeneous quadratic form in the *residual velocities* $(U - U_0)$, $(V - V_0)$, and $(W - W_0)$. In the analysis it is assumed that the *coefficients of the velocity ellipsoid* (a, b, c, f, g, and h), the motions of the *local centroids* (U_0, V_0, and W_0), and the density function (σ) are all continuous functions of position and time arbitrary in the first instance.

The general dynamical problem is formulated in Part IX. It is shown here that the mathematical problem reduces to a consideration of three groups of equations: (i) a set of ten simultaneous partial differential equations involving the derivatives with respect to the space co-ordinates of the coefficients of the velocity ellipsoid only; (ii) a set of six equations which determine the motions of the local centroids in terms of the time derivatives of the coefficients of the velocity ellipsoid and explicitly in their dependence on the space co-ordinates; (iii) a set of four equations which leads to six other *integrability conditions;* these six integrability conditions can be expressed as two vector equations (Eqs. [958] and [960]).

Part IX further contains the explicit solutions of the first two sets of equations in a Cartesian system of co-ordinates. The general solution of the sixteen equations is seen to involve twenty arbitrary functions of time and six constants of integration.

In Part X the two-dimensional problem is considered in polar co-ordinates. The complete enumeration of stellar systems in nonsteady states which have a circularly symmetrical potential function $\mathfrak{B}$ is made. The most general form for $\mathfrak{B}$ is found to be given by

$$\frac{\partial \mathfrak{B}}{\partial \bar{\omega}} = -\frac{\ddot{\phi}}{\phi}\,\bar{\omega} + \frac{1}{\phi^3}\,F\left(\frac{\bar{\omega}}{\phi}\right) , \qquad \text{(ii)}$$

where ϕ is an arbitrary function of time and F an arbitrary function of the argument specified. When

$$F\left(\frac{\bar{\omega}}{\phi}\right) = \text{constant}\,\frac{\phi^2}{\bar{\omega}^2} , \qquad \text{(iii)}$$

a "critical" case arises for which the solutions for a, b, h, U_0, and V_0 admit of a greater degree of freedom than in the general case (ii). The critical case corresponds to the gravitational field arising from a central mass together with a uniform distribution of

* This is a continuation of the author's earlier paper on "The Dynamics of Stellar Systems. I–VIII" (*Ap. J.*, **90**, 1–154, 1939). In this paper (Parts IX–XIV) the numbering of the sections, equations, footnotes, and figures is continued from the earlier paper.

mass. The possible evolutionary significance of this solution for our galactic system is further examined (§ 53). An interesting case which the analysis discloses is that in which the potential arises from the superposition of a quasi-elastic and an inverse-cube field of force. The general case of a quasi-elastic field of force is also considered. The explicit forms which the solutions for a, b, h, U_0, and V_0 take for the various cases are given.

In Part XI the enumeration of stellar systems with a spherically symmetrical potential $\mathfrak{B}$ is made (§ 58). The results obtained correspond to a three-dimensional generalization of those contained in Part X. Examples of axially symmetrical systems are also considered. It appears that the most general form for $\mathfrak{B}$ consistent with the integrability conditions is

$$\mathfrak{B} = -\frac{\ddot{\phi}}{2\phi}(x^2+y^2+z^2) + \frac{1}{\phi^2}\mathfrak{B}^*\left(\frac{\varpi}{\phi}, \frac{z}{\phi}\right) + \mathfrak{B}_0(t)\,, \tag{iv}$$

where ϕ is an arbitrary function of time and $\mathfrak{B}^*$ and $\mathfrak{B}_0$ are arbitrary functions of the arguments specified.

In Part XII the general theory of stellar systems for which the equipotential surfaces are concentric ellipsoids is worked out under nonsteady-state conditions. The mathematical problem is one of discussing the six integrability conditions for the case when $\mathfrak{B}$ has the form

$$\mathfrak{B} = \tfrac{1}{2}[\alpha_1(t)x^2 + \alpha_2(t)y^2 + \alpha_3(t)z^2]\,, \tag{v}$$

where, as indicated, α_1, α_2, and α_3 are functions of time only, arbitrary in the first instance. The case of spheroidal systems ($\alpha_1 \equiv \alpha_2$) is studied in some detail. The consideration of this case leads to the following differential equation between α_1 and α_3:

$$3\,|\alpha_3 - \alpha_1|^{1/4}\frac{d^2}{dt^2}\frac{1}{|\alpha_3 - \alpha_1|^{1/4}} + 4\alpha_1 - \alpha_3 = 0\,. \tag{vi}$$

The foregoing equation enables the study of the *stability* of the configurations ("E7") along the line $4\alpha_1 = \alpha_3$ in the (α_1, α_3) plane. It is found that these configurations possess both stable and unstable modes of vibration (see Fig. 6). Equation (vi) enables, further, the study of the evolution of "*infinitely flat*" *systems* (i.e., $\alpha_1/\alpha_3 \to 0$) and leads to the following law for the variation of the density of the spheroid with time:

$$\frac{\rho}{\rho(0)} = \frac{1}{\left[1 + \frac{\alpha_3(0)}{3}(t - t_0)^2\right]^2}\,. \tag{vii}$$

The law (vii) corresponds to a relatively rapid rate of evolution of the system. *Homologous evolution* (i.e., when α_3/α_1 = constant) of spheroidal systems is also considered. The results obtained by these methods have a bearing on questions concerning evolution along the sequence of elliptical nebulae. Particular emphasis is placed here on the circumstance that "globular systems are relatively rare as compared with lenticular systems [E7] and that numbers increase along the sequence with increasing ellipticity" (Hubble).

In Part XII the case when α_1, α_2, and α_3 are constants is also considered. In the general case ($\alpha_1 \neq \alpha_2 \neq \alpha_3$ and excluding certain special cases) the solution is found to involve twenty-one constants of integration.

In Part XIII the general theory of stellar systems in nonsteady states and described by a *spherical distribution* of the residual velocities is considered; in other words, it is assumed that

$$\Psi \equiv \Psi\{a[(U - U_0)^2 + (V - V_0)^2 + (W - W_0)^2] + \sigma\}\,, \tag{viii}$$

where a, U_0, V_0, W_0, and σ are all functions of position and time, arbitrary in the first instance. The solution to this problem is found to hinge on a single nonhomogeneous

partial differential equation for $\mathfrak{B}$ (Eq. [1596]). The general discussion of this equation for $\mathfrak{B}$ shows that

$$\sigma \equiv \sigma\{(\xi - \xi_0)^2 + (\eta - \eta_0)^2 ; \vartheta + \beta\iota ; \zeta - \zeta_0\} , \tag{ix}$$

where $a = \phi^2(t)$ and

$$x = \xi\phi ; \qquad y = \eta\phi ; \qquad z = \zeta\phi ; \qquad \iota = \int_{t_0}^{t} \frac{dt}{\phi^2} ; \tag{x}$$

further, ξ_0, η_0, and ζ_0 are certain functions of ι and

$$\vartheta = \tan^{-1} \frac{\eta - \eta_0}{\xi - \xi_0} . \tag{xi}$$

The (ξ_0, η_0) locus can, depending on circumstances, be either (i) a point, (ii) an ellipse (and the degenerate cases arising from an ellipse), (iii) an Archimedean spiral (or a generalization of it according to equation [1732]), (iv) a straight line described uniformly with respect to ι, or, finally, (v) a hyperbola. The further discussion of the density function (ix) leads to an interpretation of the spiral structure in nebulae (§ 75). The method of interpreting consists in following the trajectories of points of constant relative density, σ. These trajectories are shown to have a wide variety of forms, including "compact" spirals, open, well-resolved spirals, interpenetrating spirals, ellipses, etc. The relation between the present theory and Lindblad's theory of spiral structure is also indicated.

Part XIV deals with the problem of specifying the circumstances under which we can regard a stellar system as consisting of several independent subsystems, each of which satisfies our fundamental kinematical postulates. This part, while it is along a direction which is somewhat different from the rest of the paper, serves to generalize in a far-reaching manner the whole basis of our theory. The generalization arises from the fact that the distribution function has now the form

$$\Psi \equiv \sum_{i=1}^{n} \Psi_i(Q_i + \sigma_i) . \tag{xii}$$

INTRODUCTION

39. The general notion of "stellar systems with differential motions" has been developed at length in §§ 1 and 2. However, in setting up the fundamental differential equations of the problem (§§ 3 and 4) it was assumed that the stellar systems considered were in steady states. In this present investigation (which develops further the methods and ideas of Parts I–VIII) we shall be concerned mainly with the study of stellar systems in *nonsteady states*.

The plan of this paper is as follows:

In Part IX the precise mathematical formulation of the dynamical problem of stellar systems in nonsteady states is given, and a fundamental set of twenty simultaneous partial differential equations is derived. The general solutions for the coefficients of the

velocity ellipsoid and the motions of the local centroids are obtained. The general solutions are shown to involve twenty arbitrary functions of time and six constants of integration.

In Part X the two-dimensional problem under nonsteady-state conditions is considered. The complete enumeration of stellar systems with circular symmetry of the gravitational potential $\mathfrak{B}$ is then undertaken. The mathematical analysis in this part is particularly important, as several differential equations are encountered here for the first time which play important roles in the subsequent developments of the theory.

In Part XI examples of three-dimensional stellar systems in nonsteady states are given. The complete enumeration of stellar systems in nonsteady states and with spherical symmetry of $\mathfrak{B}$ is made. Special cases of stellar systems having an axial symmetry and in nonsteady states are also treated.

In Part XII the general theory of stellar systems for which the equipotential surfaces are concentric ellipsoids is worked out under the conditions of a nonsteady state. The evolution of spheroidal systems is studied in particular detail and leads to conclusions which appear to have significance in interpreting the sequence of elliptical nebulae (§ 67).

In Part XIII the general theory of stellar systems described by an arbitrary spherical distribution of the residual velocities is considered. The solution of this problem is shown to lead to a new theory of spiral structure in nebulae (§ 75). While the "elementary" parts of this new theory have a certain similarity with Lindblad's theory of spiral structure, the general theory appears to go considerably beyond Lindblad's work in interpreting open, barred, and peculiar spirals.

Part XIV deals with the problem of specifying the circumstances under which we can regard a stellar system as consisting of several independent subsystems, each of which satisfies our fundamental kinematical postulates. This part is therefore in a direction which is somewhat different from the rest of the paper, but it serves to generalize in a far-reaching manner the whole basis of our theory.

Parts XV and XVI (now under preparation) contain further developments of the theory.

IX. THE GENERAL THEORY OF STELLAR SYSTEMS IN NONSTEADY STATES

40. *The general formulation of the dynamical problem.*—As in § 2, we shall formulate the dynamical problem in a general orthogonal curvilinear system of co-ordinates (λ, μ, ν). Further, we shall denote the components of the velocity of a star at (λ, μ, ν) along the principal directions at (λ, μ, ν) by Λ, M, and N. We then have

$$\Lambda = P\dot{\lambda} \; ; \qquad \mathrm{M} = Q\dot{\mu} \; ; \qquad \mathrm{N} = R\dot{\nu} \, , \tag{925}$$

where P, Q, and R are certain functions of λ, μ, and ν, determined by the transformation required to change from a Cartesian system of co-ordinates (x, y, z) to the chosen system of co-ordinates (see Eq. [8]).

The kinematical characteristics which we attribute to a "stellar system" as the basis for our dynamical theory can be summarized in the form of three fundamental postulates:

Our first assumption is:

I. *At any given point* (λ, μ, ν) *we can define uniquely a local standard of rest which is a continuous function of position and time.*

Let Λ_0, M_0, and N_0 denote the components of the motion of the local centroid at (λ, μ, ν) along the principal directions at (λ, μ, ν). According to our assumption, Λ_0, M_0, and N_0 are continuous functions of λ, μ, and ν and of the time t.

The components of the residual motion of a star at (λ, μ, ν) along the principal directions are clearly

$$\Lambda - \Lambda_0 \; ; \qquad \mathrm{M} - \mathrm{M}_0 \; ; \qquad \mathrm{N} - \mathrm{N}_0 \, . \tag{926}$$

If we denote the distribution function by $\Psi(\lambda, \mu, \nu; \Lambda, \mathrm{M}, \mathrm{N}; t)$, then the number of stars $d\mathfrak{N}$ at (λ, μ, ν) with space co-ordinates in the range $(\lambda, \lambda + d\lambda; \mu, \mu + d\mu; \nu, \nu + d\nu)$ and the velocity coordinates in the range $(\Lambda, \Lambda + d\Lambda; \mathrm{M}, \mathrm{M} + d\mathrm{M}; \mathrm{N}, \mathrm{N} + d\mathrm{N})$ at time t is given by

$$d\mathfrak{N} = \Psi(\lambda, \mu, \nu; \Lambda, \mathrm{M}, \mathrm{N}; t) PQR d\lambda d\mu d\nu d\Lambda d\mathrm{M} d\mathrm{N} \, . \tag{927}$$

Our second assumption is:

II. *The distribution function* $\Psi(\lambda, \mu, \nu; \Lambda, M, N; t)$ *is of the generalized Schwarzschild type, i.e.,*

$$\Psi(\lambda, \mu, \nu; \Lambda, M, N; t) \equiv \Psi(Q + \sigma) , \tag{928}$$

where Q *stands for*

$$\left.\begin{aligned} Q = {} & a(\Lambda - \Lambda_0)^2 + b(M - M_0)^2 + c(N - N_0)^2 \\ & + 2f(M - M_0)(N - N_0) + 2g(N - N_0)(\Lambda - \Lambda_0) \\ & + 2h(\Lambda - \Lambda_0)(M - M_0) ; \end{aligned}\right\} \tag{929}$$

further, the coefficients of the velocity ellipsoid a, b, c, f, g, *and* h *and the function* σ *are all continuous functions of position* λ, μ, *and* ν *and of the time,* t.

Our third assumption is:

III. *The motions of the individual stars are governed by a potential function* $\mathfrak{B}(\lambda, \mu, \nu; t)$ *per unit mass.*

This third assumption implies that the distribution function Ψ satisfies the equation of continuity

$$\frac{\partial\Psi}{\partial t} + \frac{\Lambda}{P}\frac{\partial\Psi}{\partial\lambda} + \frac{M}{Q}\frac{\partial\Psi}{\partial\mu} + \frac{N}{R}\frac{\partial\Psi}{\partial\nu} + \dot{\Lambda}\frac{\partial\Psi}{\partial\Lambda} + \dot{M}\frac{\partial\Psi}{\partial M} + \dot{N}\frac{\partial\Psi}{\partial N} = 0 , \tag{930}$$

where the accelerations $\dot{\Lambda}$, $\dot{M}$, and $\dot{N}$ are to be expressed in terms of λ, μ, ν, Λ, M, N, and t by using the equations of motion (see § 41, below).

The dynamical problem is: *Under what circumstances will the equation of continuity (930), regarded as a partial differential equation for* Ψ, *admit of a solution of the form (928)?*

41. *The equations of the problem.*—In order to set up the fundamental differential equations of the problem, we should first express the accelerations $\dot{\Lambda}$, $\dot{M}$, and $\dot{N}$ occurring in the equation of continuity (930) in terms of the space and the velocity co-ordinates and the time, t. This can be done by using the equations of motion in

the forms given in equations (27), (28), and (29). We obtain in this manner (cf. Eq. [30])

$$\left.\begin{aligned}&\frac{\partial\Psi}{\partial t}+\frac{1}{P}\left\{\Lambda\frac{\partial\Psi}{\partial\lambda}+\frac{1}{2}\left[\frac{\mathrm{M}^2}{Q^2}\frac{\partial Q^2}{\partial\lambda}+\frac{\mathrm{N}^2}{R^2}\frac{\partial R^2}{\partial\lambda}-\frac{\Lambda\mathrm{M}}{PQ}\frac{\partial P^2}{\partial\mu}-\frac{\Lambda\mathrm{N}}{PR}\frac{\partial P^2}{\partial\nu}-2\frac{\partial\mathfrak{V}}{\partial\lambda}\right]\frac{\partial\Psi}{\partial\Lambda}\right\}\\&\quad+\frac{1}{Q}\left\{\mathrm{M}\frac{\partial\Psi}{\partial\mu}+\frac{1}{2}\left[\frac{\mathrm{N}^2}{R^2}\frac{\partial R^2}{\partial\mu}+\frac{\Lambda^2}{P^2}\frac{\partial P^2}{\partial\mu}-\frac{\mathrm{MN}}{QR}\frac{\partial Q^2}{\partial\nu}-\frac{\mathrm{M}\Lambda}{QP}\frac{\partial Q^2}{\partial\lambda}-2\frac{\partial\mathfrak{V}}{\partial\mu}\right]\frac{\partial\Psi}{\partial\mathrm{M}}\right\}\\&\quad+\frac{1}{R}\left\{\mathrm{N}\frac{\partial\Psi}{\partial\nu}+\frac{1}{2}\left[\frac{\Lambda^2}{P^2}\frac{\partial P^2}{\partial\nu}+\frac{\mathrm{M}^2}{Q^2}\frac{\partial Q^2}{\partial\nu}-\frac{\mathrm{N}\Lambda}{RP}\frac{\partial R^2}{\partial\lambda}-\frac{\mathrm{NM}}{RQ}\frac{\partial R^2}{\partial\mu}-2\frac{\partial\mathfrak{V}}{\partial\nu}\right]\frac{\partial\Psi}{\partial\mathrm{N}}\right\}=0.\end{aligned}\right\}\quad(931)$$

This is the partial differential equation for Ψ which we shall have to consider.

For a stellar system having the kinematical characteristics we have assumed (§ 40) that the distribution function Ψ must have the form (928). Substituting this form for Ψ in (931) and dividing throughout by $d\Psi/d(Q+\sigma)$, we obtain (cf. Eq. [41])

$$\left.\begin{aligned}&\frac{\partial\sigma}{\partial t}+\Lambda^2\frac{\partial a}{\partial t}+\mathrm{M}^2\frac{\partial b}{\partial t}+\mathrm{N}^2\frac{\partial c}{\partial t}+2\mathrm{MN}\frac{\partial f}{\partial t}\\&\qquad+2\mathrm{N}\Lambda\frac{\partial g}{\partial t}+2\Lambda\mathrm{M}\frac{\partial h}{\partial t}-2\Lambda\frac{\partial\Delta_1}{\partial t}-2\mathrm{M}\frac{\partial\Delta_2}{\partial t}-2\mathrm{N}\frac{\partial\Delta_3}{\partial t}+\frac{\partial Q_0}{\partial t}\\&\quad+\frac{\Lambda}{P}\left[\frac{\partial\sigma}{\partial\lambda}+\Lambda^2\frac{\partial a}{\partial\lambda}+\mathrm{M}^2\frac{\partial b}{\partial\lambda}+\mathrm{N}^2\frac{\partial c}{\partial\lambda}+2\mathrm{MN}\frac{\partial f}{\partial\lambda}\right.\\&\qquad\left.+2\mathrm{N}\Lambda\frac{\partial g}{\partial\lambda}+2\Lambda\mathrm{M}\frac{\partial h}{\partial\lambda}-2\Lambda\frac{\partial\Delta_1}{\partial\lambda}-2\mathrm{M}\frac{\partial\Delta_2}{\partial\lambda}-2\mathrm{N}\frac{\partial\Delta_3}{\partial\lambda}+\frac{\partial Q_0}{\partial\lambda}\right]\\&\quad+\frac{\mathrm{M}}{Q}\left[\frac{\partial\sigma}{\partial\mu}+\Lambda^2\frac{\partial a}{\partial\mu}+\mathrm{M}^2\frac{\partial b}{\partial\mu}+\mathrm{N}^2\frac{\partial c}{\partial\mu}+2\mathrm{MN}\frac{\partial f}{\partial\mu}\right.\\&\qquad\left.+2\mathrm{N}\Lambda\frac{\partial g}{\partial\mu}+2\Lambda\mathrm{M}\frac{\partial h}{\partial\mu}-2\Lambda\frac{\partial\Delta_1}{\partial\mu}-2\mathrm{M}\frac{\partial\Delta_2}{\partial\mu}-2\mathrm{N}\frac{\partial\Delta_3}{\partial\mu}+\frac{\partial Q_0}{\partial\mu}\right]\\&\quad+\frac{\mathrm{N}}{R}\left[\frac{\partial\sigma}{\partial\nu}+\Lambda^2\frac{\partial a}{\partial\nu}+\mathrm{M}^2\frac{\partial b}{\partial\nu}+\mathrm{N}^2\frac{\partial c}{\partial\nu}+2\mathrm{MN}\frac{\partial f}{\partial\nu}\right.\\&\qquad\left.+2\mathrm{N}\Lambda\frac{\partial g}{\partial\nu}+2\Lambda\mathrm{M}\frac{\partial h}{\partial\nu}-2\Lambda\frac{\partial\Delta_1}{\partial\nu}-2\mathrm{M}\frac{\partial\Delta_2}{\partial\nu}-2\mathrm{N}\frac{\partial\Delta_3}{\partial\nu}+\frac{\partial Q_0}{\partial\nu}\right]\\&\quad+\frac{1}{P}\left[\frac{\mathrm{M}^2}{Q^2}\frac{\partial Q^2}{\partial\lambda}+\frac{\mathrm{N}^2}{R^2}\frac{\partial R^2}{\partial\lambda}-\frac{\Lambda\mathrm{M}}{PQ}\frac{\partial P^2}{\partial\mu}-\frac{\Lambda\mathrm{N}}{PR}\frac{\partial P^2}{\partial\nu}-2\frac{\partial\mathfrak{V}}{\partial\lambda}\right][a\Lambda+h\mathrm{M}+g\mathrm{N}-\Delta_1]\\&\quad+\frac{1}{Q}\left[\frac{\mathrm{N}^2}{R^2}\frac{\partial R^2}{\partial\mu}+\frac{\Lambda^2}{P^2}\frac{\partial P^2}{\partial\mu}-\frac{\mathrm{MN}}{QR}\frac{\partial Q^2}{\partial\nu}-\frac{\mathrm{M}\Lambda}{QP}\frac{\partial Q^2}{\partial\lambda}-2\frac{\partial\mathfrak{V}}{\partial\mu}\right][h\Lambda+b\mathrm{M}+f\mathrm{N}-\Delta_2]\\&\quad+\frac{1}{R}\left[\frac{\Lambda^2}{P^2}\frac{\partial P^2}{\partial\nu}+\frac{\mathrm{M}^2}{Q^2}\frac{\partial Q^2}{\partial\nu}-\frac{\mathrm{N}\Lambda}{RP}\frac{\partial R^2}{\partial\lambda}-\frac{\mathrm{NM}}{RQ}\frac{\partial R^2}{\partial\mu}-2\frac{\partial\mathfrak{V}}{\partial\nu}\right][g\Lambda+f\mathrm{M}+c\mathrm{N}-\Delta_3]\\&=0,\end{aligned}\right\}\quad(932)$$

where Δ_1, Δ_2, and Δ_3 are defined by

$$\begin{pmatrix} a & h & g \\ h & b & f \\ g & f & c \end{pmatrix}\begin{pmatrix} \Lambda_0 \\ M_0 \\ N_0 \end{pmatrix} = \begin{pmatrix} \Delta_1 \\ \Delta_2 \\ \Delta_3 \end{pmatrix} \tag{933}$$

and

$$Q_0 = a\Lambda_0^2 + bM_0^2 + cN_0^2 + 2fM_0N_0 + 2gN_0\Lambda_0 + 2h\Lambda_0M_0 \,. \tag{934}$$

Equation (932) (which contains one hundred and four terms) is seen to be a polynomial of third degree in Λ, M, and N. Hence, the coefficients of the different power combinations of Λ, M, and N must vanish separately. Thus, equating the coefficients of Λ^3, M^3, N^3, Λ^2M, Λ^2N, $M^2\Lambda$, M^2N, $N^2\Lambda$, N^2M, ΛMN, Λ^2, M^2, N^2, ΛM, MN, $N\Lambda$, Λ, M, N, and the constant terms, we obtain, respectively,

$$\left.\begin{aligned}
&\frac{1}{P}\frac{\partial a}{\partial\lambda}+\frac{g}{RP^2}\frac{\partial P^2}{\partial\nu}+\frac{h}{QP^2}\frac{\partial P^2}{\partial\mu}=0\,, && \text{(i)}\\
&\frac{1}{Q}\frac{\partial b}{\partial\mu}+\frac{h}{PQ^2}\frac{\partial Q^2}{\partial\lambda}+\frac{f}{RQ^2}\frac{\partial Q^2}{\partial\nu}=0\,, && \text{(ii)}\\
&\frac{1}{R}\frac{\partial c}{\partial\nu}+\frac{f}{QR^2}\frac{\partial R^2}{\partial\mu}+\frac{g}{PR^2}\frac{\partial R^2}{\partial\lambda}=0\,, && \text{(iii)}\\
&\frac{2}{P}\frac{\partial h}{\partial\lambda}+\frac{1}{Q}\frac{\partial a}{\partial\mu}-(a-b)\frac{1}{QP^2}\frac{\partial P^2}{\partial\mu}+\frac{f}{RP^2}\frac{\partial P^2}{\partial\nu}-\frac{h}{PQ^2}\frac{\partial Q^2}{\partial\lambda}=0\,, && \text{(iv)}\\
&\frac{2}{P}\frac{\partial g}{\partial\lambda}+\frac{1}{R}\frac{\partial a}{\partial\nu}-(a-c)\frac{1}{RP^2}\frac{\partial P^2}{\partial\nu}+\frac{f}{QP^2}\frac{\partial P^2}{\partial\mu}-\frac{g}{PR^2}\frac{\partial R^2}{\partial\lambda}=0\,, && \text{(v)}\\
&\frac{2}{Q}\frac{\partial h}{\partial\mu}+\frac{1}{P}\frac{\partial b}{\partial\lambda}-(b-a)\frac{1}{PQ^2}\frac{\partial Q^2}{\partial\lambda}+\frac{g}{RQ^2}\frac{\partial Q^2}{\partial\nu}-\frac{h}{QP^2}\frac{\partial P^2}{\partial\mu}=0\,, && \text{(vi)}\\
&\frac{2}{Q}\frac{\partial f}{\partial\mu}+\frac{1}{R}\frac{\partial b}{\partial\nu}-(b-c)\frac{1}{RQ^2}\frac{\partial Q^2}{\partial\nu}+\frac{g}{PQ^2}\frac{\partial Q^2}{\partial\lambda}-\frac{f}{QR^2}\frac{\partial R^2}{\partial\mu}=0\,, && \text{(vii)}\\
&\frac{2}{R}\frac{\partial g}{\partial\nu}+\frac{1}{P}\frac{\partial c}{\partial\lambda}-(c-a)\frac{1}{PR^2}\frac{\partial R^2}{\partial\lambda}+\frac{h}{QR^2}\frac{\partial R^2}{\partial\mu}-\frac{g}{RP^2}\frac{\partial P^2}{\partial\nu}=0\,, && \text{(viii)}\\
&\frac{2}{R}\frac{\partial f}{\partial\nu}+\frac{1}{Q}\frac{\partial c}{\partial\mu}-(c-b)\frac{1}{QR^2}\frac{\partial R^2}{\partial\mu}+\frac{h}{PR^2}\frac{\partial R^2}{\partial\lambda}-\frac{f}{RQ^2}\frac{\partial Q^2}{\partial\nu}=0\,, && \text{(ix)}\\
&\left.\begin{aligned}&\frac{g}{QP^2}\frac{\partial P^2}{\partial\mu}+\frac{h}{RP^2}\frac{\partial P^2}{\partial\nu}+\frac{h}{RQ^2}\frac{\partial Q^2}{\partial\nu}+\frac{f}{PQ^2}\frac{\partial Q^2}{\partial\lambda}+\frac{f}{PR^2}\frac{\partial R^2}{\partial\lambda}\\
&\qquad+\frac{g}{QR^2}\frac{\partial R^2}{\partial\mu}-2\left(\frac{1}{P}\frac{\partial f}{\partial\lambda}+\frac{1}{Q}\frac{\partial g}{\partial\mu}+\frac{1}{R}\frac{\partial h}{\partial\nu}\right)=0\,,\end{aligned}\right\} && \text{(x)}
\end{aligned}\right\}\;(I_8)$$

$$\left.\begin{aligned}
&\frac{2}{P}\frac{\partial\Delta_1}{\partial\lambda}+\frac{\Delta_2}{QP^2}\frac{\partial P^2}{\partial\mu}+\frac{\Delta_3}{RP^2}\frac{\partial P^2}{\partial\nu}=\frac{\partial a}{\partial t}, && \text{(i)}\\
&\frac{2}{Q}\frac{\partial\Delta_2}{\partial\mu}+\frac{\Delta_3}{RQ^2}\frac{\partial Q^2}{\partial\nu}+\frac{\Delta_1}{PQ^2}\frac{\partial Q^2}{\partial\lambda}=\frac{\partial b}{\partial t}, && \text{(ii)}\\
&\frac{2}{R}\frac{\partial\Delta_3}{\partial\nu}+\frac{\Delta_1}{PR^2}\frac{\partial R^2}{\partial\lambda}+\frac{\Delta_2}{QR^2}\frac{\partial R^2}{\partial\mu}=\frac{\partial c}{\partial t}, && \text{(iii)}\\
&\frac{2}{P}\frac{\partial\Delta_2}{\partial\lambda}+\frac{2}{Q}\frac{\partial\Delta_1}{\partial\mu}-\frac{\Delta_1}{QP^2}\frac{\partial P^2}{\partial\mu}-\frac{\Delta_2}{PQ^2}\frac{\partial Q^2}{\partial\lambda}=2\frac{\partial h}{\partial t}, && \text{(iv)}\\
&\frac{2}{Q}\frac{\partial\Delta_3}{\partial\mu}+\frac{2}{R}\frac{\partial\Delta_2}{\partial\nu}-\frac{\Delta_2}{RQ^2}\frac{\partial Q^2}{\partial\nu}-\frac{\Delta_3}{QR^2}\frac{\partial R^2}{\partial\mu}=2\frac{\partial f}{\partial t}, && \text{(v)}\\
&\frac{2}{R}\frac{\partial\Delta_1}{\partial\nu}+\frac{2}{P}\frac{\partial\Delta_3}{\partial\lambda}-\frac{\Delta_3}{PR^2}\frac{\partial R^2}{\partial\lambda}-\frac{\Delta_1}{RP^2}\frac{\partial P^2}{\partial\nu}=2\frac{\partial g}{\partial t}, && \text{(vi)}
\end{aligned}\right\} \quad (\mathrm{II}_8)$$

and

$$\left.\begin{aligned}
&\frac{a}{P}\frac{\partial\mathfrak{B}}{\partial\lambda}+\frac{h}{Q}\frac{\partial\mathfrak{B}}{\partial\mu}+\frac{g}{R}\frac{\partial\mathfrak{B}}{\partial\nu}+\frac{\partial\Delta_1}{\partial t}=-\frac{1}{2P}\frac{\partial\chi}{\partial\lambda}, && \text{(i)}\\
&\frac{h}{P}\frac{\partial\mathfrak{B}}{\partial\lambda}+\frac{b}{Q}\frac{\partial\mathfrak{B}}{\partial\mu}+\frac{f}{R}\frac{\partial\mathfrak{B}}{\partial\nu}+\frac{\partial\Delta_2}{\partial t}=-\frac{1}{2Q}\frac{\partial\chi}{\partial\mu}, && \text{(ii)}\\
&\frac{g}{P}\frac{\partial\mathfrak{B}}{\partial\lambda}+\frac{f}{Q}\frac{\partial\mathfrak{B}}{\partial\mu}+\frac{c}{R}\frac{\partial\mathfrak{B}}{\partial\nu}+\frac{\partial\Delta_3}{\partial t}=-\frac{1}{2R}\frac{\partial\chi}{\partial\nu}, && \text{(iii)}\\
&\frac{\Delta_1}{P}\frac{\partial\mathfrak{B}}{\partial\lambda}+\frac{\Delta_2}{Q}\frac{\partial\mathfrak{B}}{\partial\mu}+\frac{\Delta_3}{R}\frac{\partial\mathfrak{B}}{\partial\nu}=+\frac{1}{2}\frac{\partial\chi}{\partial t}. && \text{(iv)}
\end{aligned}\right\} \quad (\mathrm{III}_8)$$

In equations (III_8) we have introduced the quantity χ defined by

$$-\chi = \mathfrak{Q}_0 + \sigma . \tag{935}$$

We now see that the twenty partial differential equations which result break up into three distinct sets of equations. The first group of ten equations involves only the coefficients of the velocity ellipsoid. Further, these equations do not involve any differentiations with respect to time and are, in fact, identical with those found in our analysis of the steady-state problem (cf. Eqs. I, Part I). Consequently, the first group of ten equations will be sufficient to determine the dependence of the coefficients of the velocity ellipsoid on

the space co-ordinates only and will leave their dependence on time unspecified.

The second group of six equations involves the Δ's and the time derivatives of the coefficients of the velocity ellipsoid. These equations, as we shall see, are sufficient to determine the dependence of the Δ's on the space co-ordinates. Further, the equations (II_8) are found to restrict the dependence of the coefficients of the velocity ellipsoid on time. Thus, if, in particular, we consider stellar systems *without* differential motions (i.e., $\Lambda_0 \equiv M_0 \equiv N_0 \equiv 0$), then the Δ's all vanish identically, and the equations (II_8) now imply that the coefficients of the velocity ellipsoid do not depend upon the time.[28]

The last group of four equations (III_8) are of a nature different from the rest and lead to six other integrability conditions (see Eqs. [958] and [960], below).

The equations (III_8) enable us to derive an interesting relation: Multiply the equations (i), (ii), and (iii) by Λ_0, M_0, and N_0, respectively, and add. Using the relations defined by (933), we thus obtain

$$\left.\begin{aligned}\frac{\Delta_1}{P}\frac{\partial\mathfrak{V}}{\partial\lambda}+\frac{\Delta_2}{Q}\frac{\partial\mathfrak{V}}{\partial\mu}+\frac{\Delta_3}{R}\frac{\partial\mathfrak{V}}{\partial\nu}+\Lambda_0\frac{\partial\Delta_1}{\partial t}+M_0\frac{\partial\Delta_2}{\partial t}+N_0\frac{\partial\Delta_3}{\partial t}&\\ +\frac{1}{2}\left(\frac{\Lambda_0}{P}\frac{\partial\chi}{\partial\lambda}+\frac{M_0}{Q}\frac{\partial\chi}{\partial\mu}+\frac{N_0}{R}\frac{\partial\chi}{\partial\nu}\right)=0\,.&\end{aligned}\right\}\quad(936)$$

Using equation (iv) of (III_8), we now obtain

$$\left.\begin{aligned}\frac{\partial\chi}{\partial t}+\frac{\Lambda_0}{P}\frac{\partial\chi}{\partial\lambda}+\frac{M_0}{Q}\frac{\partial\chi}{\partial\mu}+\frac{N_0}{R}\frac{\partial\chi}{\partial\nu}&\\ +2\left(\Lambda_0\frac{\partial\Delta_1}{\partial t}+M_0\frac{\partial\Delta_2}{\partial t}+N_0\frac{\partial\Delta_3}{\partial t}\right)=0\,.&\end{aligned}\right\}\quad(937)$$

42. *The equations for the two-dimensional problem.*—The fundamental differential equations for the two-dimensional case can be obtained from (I_8), (II_8), and (III_8) by setting $c=f=g=0$ and

[28] It is thus apparent that the consideration of nonsteady states is nontrivial only for the case of stellar systems *with* differential motions.

further ignoring all the terms involving differentiations with respect to ν. We have

$$\left.\begin{aligned}
&\frac{1}{P}\frac{\partial a}{\partial \lambda}+\frac{h}{QP^2}\frac{\partial P^2}{\partial \mu}=0\,, && \text{(i)}\\
&\frac{1}{Q}\frac{\partial b}{\partial \mu}+\frac{h}{PQ^2}\frac{\partial Q^2}{\partial \lambda}=0\,, && \text{(ii)}\\
&\frac{2}{P}\frac{\partial h}{\partial \lambda}+\frac{1}{Q}\frac{\partial a}{\partial \mu}-(a-b)\frac{1}{QP^2}\frac{\partial P^2}{\partial \mu}-\frac{h}{PQ^2}\frac{\partial Q^2}{\partial \lambda}=0\,, && \text{(iii)}\\
&\frac{2}{Q}\frac{\partial h}{\partial \mu}+\frac{1}{P}\frac{\partial b}{\partial \lambda}-(b-a)\frac{1}{PQ^2}\frac{\partial Q^2}{\partial \lambda}-\frac{h}{QP^2}\frac{\partial P^2}{\partial \mu}=0\,, && \text{(iv)}
\end{aligned}\right\}\quad (\mathrm{I}_9)$$

$$\left.\begin{aligned}
&\frac{2}{P}\frac{\partial \Delta_1}{\partial \lambda}+\frac{\Delta_2}{QP^2}\frac{\partial P^2}{\partial \mu}=\frac{\partial a}{\partial t}\,, && \text{(i)}\\
&\frac{2}{Q}\frac{\partial \Delta_2}{\partial \mu}+\frac{\Delta_1}{PQ^2}\frac{\partial Q^2}{\partial \lambda}=\frac{\partial b}{\partial t}\,, && \text{(ii)}\\
&\frac{2}{P}\frac{\partial \Delta_2}{\partial \lambda}+\frac{2}{Q}\frac{\partial \Delta_1}{\partial \mu}-\frac{\Delta_1}{QP^2}\frac{\partial P^2}{\partial \mu}-\frac{\Delta_2}{PQ^2}\frac{\partial Q^2}{\partial \lambda}=2\frac{\partial h}{\partial t}\,, && \text{(iii)}
\end{aligned}\right\}\quad (\mathrm{II}_9)$$

and

$$\left.\begin{aligned}
&\frac{a}{P}\frac{\partial \mathfrak{B}}{\partial \lambda}+\frac{h}{Q}\frac{\partial \mathfrak{B}}{\partial \mu}+\frac{\partial \Delta_1}{\partial t}=-\frac{1}{2P}\frac{\partial \chi}{\partial \lambda}\,, && \text{(i)}\\
&\frac{h}{P}\frac{\partial \mathfrak{B}}{\partial \lambda}+\frac{b}{Q}\frac{\partial \mathfrak{B}}{\partial \mu}+\frac{\partial \Delta_2}{\partial t}=-\frac{1}{2Q}\frac{\partial \chi}{\partial \mu}\,, && \text{(ii)}\\
&\frac{\Delta_1}{P}\frac{\partial \mathfrak{B}}{\partial \lambda}+\frac{\Delta_2}{Q}\frac{\partial \mathfrak{B}}{\partial \mu}=+\frac{1}{2}\frac{\partial \chi}{\partial t}\,. && \text{(iii)}
\end{aligned}\right\}\quad (\mathrm{III}_9)$$

In the above equations Δ_1, Δ_2, and χ have the following meanings:

$$\Delta_1 = a\Lambda_0 + hM_0\,; \qquad \Delta_2 = h\Lambda_0 + bM_0\,, \tag{938}$$

and

$$-\chi = \mathfrak{Q}_0 + \sigma\,, \tag{939}$$

where

$$Q_0 = a\Lambda_0^2 + 2h\Lambda_0 M_0 + bM_0^2 . \tag{940}$$

43. *The fundamental equations in Cartesian co-ordinates.*—For the fundamental frame of reference we shall choose a Cartesian system of co-ordinates (x, y, z). For such a system $P = Q = R = 1$, and the differential equations (I_8), (II_8), and (III_8) now become

$$\left.\begin{array}{llll}
\dfrac{\partial a}{\partial x} = 0 , \quad \text{(i)} ; & \dfrac{\partial b}{\partial y} = 0 , \quad \text{(ii)} ; & \dfrac{\partial c}{\partial z} = 0 , \quad \text{(iii)} \\[2ex]
2\dfrac{\partial h}{\partial x} + \dfrac{\partial a}{\partial y} = 0 , & \text{(iv)} ; & 2\dfrac{\partial g}{\partial x} + \dfrac{\partial a}{\partial z} = 0 , & \text{(v)} \\[2ex]
2\dfrac{\partial h}{\partial y} + \dfrac{\partial b}{\partial x} = 0 , & \text{(vi)} ; & 2\dfrac{\partial f}{\partial y} + \dfrac{\partial b}{\partial z} = 0 , & \text{(vii)} \\[2ex]
2\dfrac{\partial g}{\partial z} + \dfrac{\partial c}{\partial x} = 0 , & \text{(viii)} ; & 2\dfrac{\partial f}{\partial z} + \dfrac{\partial c}{\partial y} = 0 , & \text{(ix)} \\[2ex]
\multicolumn{4}{c}{\dfrac{\partial f}{\partial x} + \dfrac{\partial g}{\partial y} + \dfrac{\partial h}{\partial z} = 0 , \quad \text{(x)}}
\end{array}\right\} \quad (I_{10})$$

$$\left.\begin{array}{llll}
\dfrac{\partial \Delta_1}{\partial x} = \dfrac{1}{2}\dfrac{\partial a}{\partial t} , & \text{(i)} ; & \dfrac{\partial \Delta_2}{\partial x} + \dfrac{\partial \Delta_1}{\partial y} = \dfrac{\partial h}{\partial t} , & \text{(iv)} \\[2ex]
\dfrac{\partial \Delta_2}{\partial y} = \dfrac{1}{2}\dfrac{\partial b}{\partial t} , & \text{(ii)} ; & \dfrac{\partial \Delta_3}{\partial y} + \dfrac{\partial \Delta_2}{\partial z} = \dfrac{\partial f}{\partial t} , & \text{(v)} \\[2ex]
\dfrac{\partial \Delta_3}{\partial z} = \dfrac{1}{2}\dfrac{\partial c}{\partial t} , & \text{(iii)} ; & \dfrac{\partial \Delta_1}{\partial z} + \dfrac{\partial \Delta_3}{\partial x} = \dfrac{\partial g}{\partial t} , & \text{(vi)}
\end{array}\right\} \quad (II_{10})$$

and

$$\left.\begin{array}{l}
\begin{pmatrix} a & h & g \\ h & b & f \\ g & f & c \end{pmatrix} \operatorname{grad} \mathfrak{B} + \dfrac{\partial \boldsymbol{\Delta}}{\partial t} = -\tfrac{1}{2} \operatorname{grad} \chi , \\[2ex]
\qquad\qquad \boldsymbol{\Delta} \cdot \operatorname{grad} \mathfrak{B} = \dfrac{1}{2}\dfrac{\partial \chi}{\partial t} .
\end{array}\right\} \quad (III_{10})$$

In equation (III_{10}), $\boldsymbol{\Delta}$ represents the vector $(\Delta_1, \Delta_2, \Delta_3)$.

44. *The solution for the coefficients of the velocity ellipsoid.*—Since the equations (I_{10}) are identical with those for the steady-state problem (cf. Eq. [I_4], Part III), we can at once write down the dependence of the coefficients of the velocity ellipsoid on the space co-ordinates (x, y, z). We have (cf. Eqs. [183]–[185] and [191]–[193])

$$\left.\begin{aligned}
a &= -2(h_{20}+h_{21}z)y - h_{40}y^2 - (a_0 + 2g_{30}z + g_{40}z^2)\,,\\
b &= -2(f_{20}+f_{21}x)z - f_{40}z^2 - (b_0 + 2h_{30}x + h_{40}x^2)\,,\\
c &= -2(g_{20}+g_{21}y)x - g_{40}x^2 - (c_0 + 2f_{30}y + f_{40}y^2)\,,\\
f &= (f_{10}+f_{11}x - h_{21}x^2) + (f_{20}+f_{21}x)y + (f_{30}+g_{21}x)z + f_{40}yz\,,\\
g &= (g_{10}+g_{11}y - f_{21}y^2) + (g_{20}+g_{21}y)z + (g_{30}+h_{21}y)x + g_{40}zx\,,\\
h &= (h_{10}+h_{11}z - g_{21}z^2) + (h_{20}+h_{21}z)x + (h_{30}+f_{21}z)y + h_{40}xy\,.
\end{aligned}\right\}\tag{941}$$

In the foregoing expressions for a, b, c, f, g, and h the quantities a_0, b_0, c_0, f_{10}, g_{10}, h_{10}, f_{20}, g_{20}, h_{20}, f_{21}, g_{21}, h_{21}, f_{30}, g_{30}, h_{30}, f_{40}, g_{40}, and h_{40} should all be regarded (*at this stage*) as arbitrary functions of time. Of the remaining three quantities, f_{11}, g_{11}, and h_{11}, two of them are again to be regarded as arbitrary functions of time, while the third has to be found from the relation (cf. Eq. [186])

$$f_{11}+g_{11}+h_{11} = 0\,. \tag{942}$$

45. *The solution for the motions of the local centroids.*—We shall now consider the six equations (II_{10}). From these equations and the nature of the dependence of the coefficients of the velocity ellipsoid on x, y, and z (see Eq. [167]) we readily obtain

$$\left.\begin{aligned}
&\frac{\partial^2\Delta_1}{\partial x^2} = 0\,; \quad \frac{\partial^3\Delta_1}{\partial y^3} = 0\,; \quad \frac{\partial^3\Delta_1}{\partial z^3} = 0\,,\\
&\frac{\partial^2\Delta_2}{\partial y^2} = 0\,; \quad \frac{\partial^3\Delta_2}{\partial z^3} = 0\,; \quad \frac{\partial^3\Delta_2}{\partial x^3} = 0\,,\\
&\frac{\partial^2\Delta_3}{\partial z^2} = 0\,; \quad \frac{\partial^3\Delta_3}{\partial x^3} = 0\,; \quad \frac{\partial^3\Delta_3}{\partial y^3} = 0\,.
\end{aligned}\right\}\tag{943}$$

In other words,

$$\left.\begin{array}{l}\Delta_1 \text{ is linear in } x \text{ and quadratic in } y \text{ and } z\,,\\ \Delta_2 \text{ is linear in } y \text{ and quadratic in } z \text{ and } x\,,\\ \Delta_3 \text{ is linear in } z \text{ and quadratic in } x \text{ and } y\,.\end{array}\right\} \tag{944}$$

On the other hand, since a, b, and c are independent of x, y, and z, respectively, we readily infer from equations (i), (ii), and (iii) that

$$\left.\begin{aligned}\Delta_1 &= \frac{1}{2}\frac{\partial a}{\partial t}x + \text{function }(y, z, t)\,,\\ \Delta_2 &= \frac{1}{2}\frac{\partial b}{\partial t}y + \text{function }(z, x, t)\,,\\ \Delta_3 &= \frac{1}{2}\frac{\partial c}{\partial t}z + \text{function }(x, y, t)\,.\end{aligned}\right\} \tag{945}$$

Combining the relations (944) and (945), we can write

$$\left.\begin{aligned}\Delta_1 &= \frac{1}{2}\frac{\partial a}{\partial t}x + \alpha_{22}y^2z^2 + \alpha_{21}y^2z + \alpha_{12}yz^2 + \alpha_{20}y^2 + \alpha_{02}z^2\\ &\qquad + \alpha_{11}yz + \beta_3 y + \gamma_2 z + \delta_1\,,\\ \Delta_2 &= \frac{1}{2}\frac{\partial b}{\partial t}y + \beta_{22}z^2x^2 + \beta_{21}z^2x + \beta_{12}zx^2 + \beta_{20}z^2 + \beta_{02}x^2\\ &\qquad + \beta_{11}zx + \beta_1 z + \gamma_3 x + \delta_2\,,\\ \Delta_3 &= \frac{1}{2}\frac{\partial c}{\partial t}z + \gamma_{22}x^2y^2 + \gamma_{21}x^2y + \gamma_{12}xy^2 + \gamma_{20}x^2 + \gamma_{02}y^2\\ &\qquad + \gamma_{11}xy + \beta_2 x + \gamma_1 y + \delta_3\,,\end{aligned}\right\} \tag{946}$$

where α_{22}, α_{21}, ... , δ_3 are all functions of time and arbitrary in the first instance. Substituting for Δ_1, Δ_2, and Δ_3 according to (946) in equations (iv), (v), and (vi), we obtain, respectively,

$$\left.\begin{aligned}
&\frac{\partial}{\partial t}(h_{10}+h_{11}z+h_{20}x+h_{30}y-g_{21}z^2+h_{21}zx+f_{21}zy+h_{40}xy)\\
&\quad=2\beta_{22}z^2x+2\alpha_{22}yz^2+(\beta_{21}+\alpha_{12})z^2+2\beta_{12}zx+2\alpha_{21}yz+(\alpha_{11}+\beta_{11})z\\
&\quad+2\beta_{02}x+2\alpha_{20}y+(\beta_3+\gamma_3)-y\frac{\partial}{\partial t}(h_{30}+h_{40}x+f_{21}z)\\
&\quad-x\frac{\partial}{\partial t}(h_{20}+h_{40}y+h_{21}z)\,,\\
&\frac{\partial}{\partial t}(f_{10}+f_{11}x+f_{20}y+f_{30}z-h_{21}x^2+f_{21}xy+g_{21}xz+f_{40}yz)\\
&\quad=2\gamma_{22}x^2y+2\beta_{22}zx^2+(\gamma_{21}+\beta_{12})x^2+2\gamma_{12}xy+2\beta_{21}zx+(\beta_{11}+\gamma_{11})x\\
&\quad+2\gamma_{02}y+2\beta_{20}z+(\beta_1+\gamma_1)-z\frac{\partial}{\partial t}(f_{30}+f_{40}y+g_{21}x)\\
&\quad-y\frac{\partial}{\partial t}(f_{20}+f_{40}z+f_{21}x)\,,\\
&\frac{\partial}{\partial t}(g_{10}+g_{11}y+g_{20}z+g_{30}x-f_{21}y^2+g_{21}yz+h_{21}yx+g_{40}zx)\\
&\quad=2\alpha_{22}y^2z+2\gamma_{22}xy^2+(\alpha_{21}+\gamma_{12})y^2+2\alpha_{12}yz+2\gamma_{21}xy+(\gamma_{11}+\alpha_{11})y\\
&\quad+2\alpha_{02}z+2\gamma_{20}x+(\beta_2+\gamma_2)-x\frac{\partial}{\partial t}(g_{30}+g_{40}z+h_{21}y)\\
&\quad-z\frac{\partial}{\partial t}(g_{20}+g_{40}x+g_{21}y)\,.
\end{aligned}\right\}\qquad(947)$$

Equating the coefficients of the different combinations of the powers of x, y, and z in the foregoing equations, we find

$$\left.\begin{aligned}
&\alpha_{22}=\beta_{22}=\gamma_{22}=0\,,\\
&\frac{df_{40}}{dt}=\frac{dg_{40}}{dt}=\frac{dh_{40}}{dt}=0\,,
\end{aligned}\right\}\qquad(948_1)$$

$$\left.\begin{aligned}
&\alpha_{20}=\frac{dh_{30}}{dt}\,;\qquad \beta_{20}=\frac{df_{30}}{dt}\,;\qquad \gamma_{20}=\frac{dg_{30}}{dt}\,,\\
&\alpha_{02}=\frac{dg_{20}}{dt}\,;\qquad \beta_{02}=\frac{dh_{20}}{dt}\,;\qquad \gamma_{02}=\frac{df_{20}}{dt}\,,
\end{aligned}\right\}\qquad(948_2)$$

$$\left.\begin{aligned}
&\beta_{21}+\alpha_{12}=-\frac{dg_{21}}{dt}\,;\qquad \alpha_{12}=\frac{dg_{21}}{dt}\,;\qquad \alpha_{21}=\frac{df_{21}}{dt}\,,\\
&\gamma_{21}+\beta_{12}=-\frac{dh_{21}}{dt}\,;\qquad \beta_{12}=\frac{dh_{21}}{dt}\,;\qquad \beta_{21}=\frac{dg_{21}}{dt}\,,\\
&\alpha_{21}+\gamma_{12}=-\frac{df_{21}}{dt}\,;\qquad \gamma_{12}=\frac{df_{21}}{dt}\,;\qquad \gamma_{21}=\frac{dh_{21}}{dt}\,,
\end{aligned}\right\}\qquad(948_3)$$

$$\beta_{11}+\gamma_{11}=\frac{df_{11}}{dt}\,;\qquad \gamma_{11}+\alpha_{11}=\frac{dg_{11}}{dt}\,;\qquad \alpha_{11}+\beta_{11}=\frac{dh_{11}}{dt}\,,\qquad(948_4)$$

and

$$\beta_1 + \gamma_1 = \frac{df_{10}}{dt}\,; \qquad \beta_2 + \gamma_2 = \frac{dg_{10}}{dt}\,; \qquad \beta_3 + \gamma_3 = \frac{dh_{10}}{dt}\,. \tag{949}$$

From equations (948_3) it readily follows that

$$\alpha_{12} = \alpha_{21} = \beta_{12} = \beta_{21} = \gamma_{12} = \gamma_{21} = 0 \tag{950}$$

and that

$$\frac{df_{21}}{dt} = \frac{dg_{21}}{dt} = \frac{dh_{21}}{dt} = 0\,. \tag{951}$$

Adding the three equations (948_4) and using (942), we find

$$\alpha_{11} + \beta_{11} + \gamma_{11} = 0\,. \tag{952}$$

We can therefore re-write the equations (948_4) in the forms

$$\alpha_{11} = -\frac{df_{11}}{dt}\,; \qquad \beta_{11} = -\frac{dg_{11}}{dt}\,; \qquad \gamma_{11} = -\frac{dh_{11}}{dt}\,. \tag{952'}$$

Finally, substituting for α_{20}, β_{20}, γ_{20}, α_{02}, β_{02}, γ_{02}, α_{11}, β_{11}, and γ_{11} according to the equations (948_2) and (952′) in our expressions for the Δ's (Eq. [946]) and after some minor rearranging of the terms, we obtain

$$\left.\begin{aligned}
\Delta_1 &= y\left(y\frac{dh_{30}}{dt} - x\frac{dh_{20}}{dt}\right) - z\left(x\frac{dg_{30}}{dt} - z\frac{dg_{20}}{dt}\right)\\
&\qquad\qquad - \frac{df_{11}}{dt}\,yz - \frac{1}{2}\frac{da_0}{dt}\,x + \beta_3 y + \gamma_2 z + \delta_1\,,\\
\Delta_2 &= z\left(z\frac{df_{30}}{dt} - y\frac{df_{20}}{dt}\right) - x\left(y\frac{dh_{30}}{dt} - x\frac{dh_{20}}{dt}\right)\\
&\qquad\qquad - \frac{dg_{11}}{dt}\,zx - \frac{1}{2}\frac{db_0}{dt}\,y + \beta_1 z + \gamma_3 x + \delta_2\,,\\
\Delta_3 &= x\left(x\frac{dg_{30}}{dt} - z\frac{dg_{20}}{dt}\right) - y\left(z\frac{df_{30}}{dt} - y\frac{df_{20}}{dt}\right)\\
&\qquad\qquad - \frac{dh_{11}}{dt}\,xy - \frac{1}{2}\frac{dc_0}{dt}\,z + \beta_2 x + \gamma_1 y + \delta_3\,.
\end{aligned}\right\} \tag{953}$$

We have further shown that

$$f_{21},\ g_{21},\ h_{21},\ f_{40},\ g_{40},\ h_{40} \text{ are all constants .} \tag{954}$$

We thus see that the solution for the coefficients of the velocity ellipsoid involve only fourteen arbitrary functions of time.[29] In addition to these fourteen functions the solution for the Δ's introduces six further arbitrary functions of time.[30]

46. *The integrability relations.*—We have now solved the sixteen differential equations ($[I_{10}]$ and $[II_{10}]$) for the coefficients of the velocity ellipsoid and the motions of the local centroids. These solutions are seen to involve twenty arbitrary functions of time and six constants of integration. But it should not be concluded that the solution for the physical problem involves this degree of arbitrariness. Indeed, the further discussion of the relations (III_{10}) will impose restrictions of a very far reaching character on the possible solutions. The mathematical problem which is presented can be formulated as follows:

It is convenient to introduce a symbol for the matrix associated with our fundamental quadratic form. Let

$$\boldsymbol{A} = \begin{pmatrix} a & h & g \\ h & b & f \\ g & f & c \end{pmatrix} . \tag{955}$$

The equations (III_{10}) can be written in the forms

$$\boldsymbol{A} \text{ grad } \mathfrak{B} + \frac{\partial \boldsymbol{\Delta}}{\partial t} = -\tfrac{1}{2} \text{ grad } \chi \tag{956}$$

and

$$\boldsymbol{\Delta} \cdot \text{grad } \mathfrak{B} = +\frac{1}{2}\frac{\partial \chi}{\partial t} , \tag{957}$$

[29] These are a_0, b_0, c_0, f_{10}, g_{10}, h_{10}, f_{20}, g_{20}, h_{20}, f_{30}, g_{30}, h_{30}, and any two of the three quantities f_{11}, g_{11}, and h_{11}. It should be pointed out in this connection that we have not so far considered the restrictions which the integrability conditions resulting from (III_{10}) will impose on these quantities (see § 46).

[30] These can be taken to be β_1, β_2, β_3, δ_1, δ_2, and δ_3. The functions γ_1, γ_2, and γ_3 are related to β_1, β_2, β_3, f_{10}, g_{10}, and h_{10} according to the relations (949).

where $\boldsymbol{\Delta}$ stands for the vector $(\delta_1, \delta_2, \delta_3)$. Taking the curl of both the sides of equation (956), we obtain

$$\text{curl } (\boldsymbol{A} \text{ grad } \mathfrak{B}) + \frac{\partial}{\partial t}(\text{curl } \boldsymbol{\Delta}) = 0 . \tag{958}$$

The vector equation (958) represents three simultaneous partial differential equations of the second order for $\mathfrak{B}$.

Again, taking the gradient of both the sides of equation (957), we have

$$\text{grad } (\boldsymbol{\Delta} \cdot \text{grad } \mathfrak{B}) = \frac{1}{2}\frac{\partial}{\partial t} \text{grad } \chi ; \tag{959}$$

or, using equation (956), we obtain

$$\text{grad } (\boldsymbol{\Delta} \cdot \text{grad } \mathfrak{B}) + \frac{\partial}{\partial t}(\boldsymbol{A} \text{ grad } \mathfrak{B}) + \frac{\partial^2 \boldsymbol{\Delta}}{\partial t^2} = 0 . \tag{960}$$

The vector equation (960) again represents three further simultaneous partial differential equations of the second order for $\mathfrak{B}$.

Now, we know the elements of the matrix $\boldsymbol{A}$ and the components of the vector $\boldsymbol{\Delta}$, apart from some arbitrary functions of time and some constants of integration. Consequently, in the six simultaneous partial differential equations of the second order in $\mathfrak{B}$ which the vector equations (958) and (960) represent, the coefficients of the various derivatives of $\mathfrak{B}$ are of known forms. It is therefore clear that there should be restrictions on these coefficients of the various derivatives of $\mathfrak{B}$ in the six equations, in order that all these partial differential equations may admit of a common solution $\mathfrak{B}$. If we could enumerate all the different circumstances under which the six differential equations (958) and (960) will admit of a common solution, we would then have solved our dynamical problem completely. Actually, it has not been found possible to make any progress toward the solution of the problem in its most general form. However, a number of special solutions have been found which seem to have a definite interest. Parts X–XIII are devoted to the consideration of these special cases which seem to arise more or less naturally. It is thought possible that this discussion of special cases may give an insight into the general structure of the equations (958) and (960).

47. *The solution for the coefficients of the velocity ellipse and the motions of the local centroids for the two-dimensional problem.*—We shall conclude our discussion of the general theory by writing down the solution for a, b, h, Δ_1, and Δ_2 appropriate for the discussion of the two-dimensional problem. These can be obtained quite readily from equations (941), (953), and (954) by setting all the coefficients which occur in the expressions for c, f, g, and Δ_3 equal to zero. We find

$$\left.\begin{aligned} a &= -a_0 - 2h_2 y - h_4 y^2 , \\ b &= -b_0 - 2h_3 x - h_4 x^2 , \\ h &= h_1 + h_2 x + h_3 y + h_4 xy , \end{aligned}\right\} \tag{961}$$

where

$$h_4 = \text{constant} \tag{962}$$

and where a_0, b_0, h_1, h_2, and h_3 are arbitrary functions of time. Further,

$$\left.\begin{aligned} \Delta_1 &= y\left(y\frac{dh_3}{dt} - x\frac{dh_2}{dt}\right) - \frac{1}{2}\frac{da_0}{dt}x + \beta_3 y + \delta_1 , \\ \Delta_2 &= -x\left(y\frac{dh_3}{dt} - x\frac{dh_2}{dt}\right) - \frac{1}{2}\frac{db_0}{dt}y + \gamma_3 x + \delta_2 , \end{aligned}\right\} \tag{963}$$

where δ_1 and δ_2 are arbitrary functions of time and

$$\beta_3 + \gamma_3 = \frac{dh_1}{dt} ; \tag{964}$$

thus, only one of the two quantities β_3 and γ_3 can be an arbitrary function of time.

X. THE TWO-DIMENSIONAL PROBLEM; STELLAR SYSTEMS WITH CIRCULAR SYMMETRY AND IN NONSTEADY STATES

48. *The solutions for* a, b, h, Δ_1, *and* Δ_2 *in polar co-ordinates.*—In § 47 we have already given the solutions for the coefficients of the velocity ellipse and for the motions of the local centroids in Cartesian co-ordinates. However, since the discussion of stellar systems with circular symmetry is most conveniently carried out in polar

co-ordinates, we shall first obtain the solutions for a, b, h, Δ_1, and Δ_2 in these co-ordinates.

As we have seen (§ 41), the solutions for the coefficients of the velocity ellipsoid (or ellipse, in the two-dimensional case) take the same forms both in the steady-state and in the nonsteady-state cases. We can therefore write down the solutions appropriate for the present case from the solutions obtained for the general three-dimensional problem in cylindrical co-ordinates (Part VII, § 29). We have (cf. Eqs. [711])

$$\left.\begin{aligned} a &= a_1 + h_5 \sin 2\theta - h_6 \cos 2\theta \,, \\ b &= a_1 - h_5 \sin 2\theta + h_6 \cos 2\theta - 2(h_3 \sin \theta - h_4 \cos \theta)\bar{\omega} + b_{20}\bar{\omega}^2 \,, \\ h &= h_5 \cos 2\theta + h_6 \sin 2\theta + (h_3 \cos \theta + h_4 \sin \theta)\bar{\omega} \,, \end{aligned}\right\} \quad (965)$$

where a_1, b_{20}, h_3, h_4, h_5, and h_6 are to be regarded (for the present) as arbitrary functions of time. Let

$$h_1 = h_3 \cos \theta + h_4 \sin \theta \,; \qquad h_2 = h_5 \cos 2\theta + h_6 \sin 2\theta \,. \quad (966)$$

In terms of h_1 and h_2 we can re-write our solution (965) as

$$\left.\begin{aligned} a &= a_1 - \frac{1}{2}\frac{\partial h_2}{\partial \theta} \,, \\ b &= a_1 + \frac{1}{2}\frac{\partial h_2}{\partial \theta} + 2\frac{\partial h_1}{\partial \theta}\bar{\omega} + b_{20}\bar{\omega}^2 \,, \\ h &= h_1\bar{\omega} + h_2 \,. \end{aligned}\right\} \quad (967)$$

We shall next proceed to obtain the solutions for Δ_1 and Δ_2. The differential equations determining these are (cf. Eqs. [II_9])

$$\left.\begin{aligned} &\frac{\partial \Delta_1}{\partial \bar{\omega}} = \frac{1}{2}\frac{\partial a}{\partial t} \,, && \text{(i)} \\ &\frac{\partial \Delta_2}{\partial \theta} + \Delta_1 = \frac{\bar{\omega}}{2}\frac{\partial b}{\partial t} \,, && \text{(ii)} \\ &\bar{\omega}\frac{\partial \Delta_2}{\partial \bar{\omega}} + \frac{\partial \Delta_1}{\partial \theta} - \Delta_2 = \bar{\omega}\frac{\partial h}{\partial t} \,. && \text{(iii)} \end{aligned}\right\} \quad (\mathrm{II}_{11})$$

Since a is independent of $\bar{\omega}$, equation (i) can be integrated to give

$$\Delta_1 = \delta_1(\theta, t) + \frac{\bar{\omega}}{2}\frac{\partial a}{\partial t}\,, \tag{968}$$

where δ_1 is independent of $\bar{\omega}$. Differentiating equation (iii) partially with respect to θ, we obtain

$$\bar{\omega}\frac{\partial^2 \Delta_2}{\partial\bar{\omega}\partial\theta} + \frac{\partial^2 \Delta_1}{\partial\theta^2} - \frac{\partial\Delta_2}{\partial\theta} = \bar{\omega}\frac{\partial^2 h}{\partial t\partial\theta}\,. \tag{969}$$

On the other hand, from equations (i) and (ii) we readily obtain

$$\frac{\partial^2 \Delta_2}{\partial\bar{\omega}\partial\theta} = \frac{1}{2}\frac{\partial}{\partial t}\left(b - a + \bar{\omega}\frac{\partial b}{\partial\bar{\omega}}\right). \tag{970}$$

Substituting from equations (ii), (968), and (970) in (969), we find

$$\frac{\partial^2\delta_1}{\partial\theta^2} + \delta_1 = \bar{\omega}\frac{\partial}{\partial t}\left(\frac{\partial h}{\partial\theta} - \frac{1}{2}\frac{\partial^2 a}{\partial\theta^2} - \frac{\bar{\omega}}{2}\frac{\partial b}{\partial\bar{\omega}}\right). \tag{971}$$

Using the expressions for a, b, and h as given by equations (965), we find that the foregoing equation can be reduced to

$$\frac{\partial^2\delta_1}{\partial\theta^2} + \delta_1 = -\bar{\omega}^3\frac{db_{20}}{dt}\,. \tag{972}$$

Since, however, δ_1 is independent of $\bar{\omega}$, we see that

$$b_{20} = \text{constant.} \tag{973}$$

From equation (972) we now obtain

$$\delta_1 = \beta_1(t)\cos\theta + \beta_2(t)\sin\theta\,, \tag{974}$$

where, as the notation implies, β_1 and β_2 are functions of time only. Combining the relations (968) and (974), we have

$$\Delta_1 = \beta_1\cos\theta + \beta_2\sin\theta + \frac{\bar{\omega}}{2}\frac{\partial}{\partial t}(h_5\sin 2\theta - h_6\cos 2\theta + a_1)\,. \tag{975}$$

Substituting the foregoing expression for Δ_1 in equation (ii), we find (remembering that b_{20} is a constant)

$$\left.\begin{aligned}\frac{\partial \Delta_2}{\partial \theta} = -\beta_1 \cos\theta - \beta_2 \sin\theta - \bar{\omega}^2 \frac{\partial}{\partial t}(h_3 \sin\theta - h_4 \cos\theta)\\ - \bar{\omega}\frac{\partial}{\partial t}(h_5 \sin 2\theta - h_6 \cos 2\theta)\,,\end{aligned}\right\} \quad (976)$$

or, after integration,

$$\left.\begin{aligned}\Delta_2 = -\beta_1 \sin\theta + \beta_2 \cos\theta + \bar{\omega}^2 \frac{\partial}{\partial t}(h_3 \cos\theta + h_4 \sin\theta)\\ + \frac{\bar{\omega}}{2}\frac{\partial}{\partial t}(h_5 \cos 2\theta + h_6 \sin 2\theta) + \delta_2(\bar{\omega}, t)\,.\end{aligned}\right\} \quad (977)$$

Finally, substituting in (ii) for Δ_1 and Δ_2 according to equations (975) and (977), we obtain

$$\bar{\omega}\frac{\partial \delta_2}{\partial \bar{\omega}} = \delta_2\,, \quad (978)$$

or

$$\delta_2 = p(t)\bar{\omega}\,, \quad (979)$$

where p is again a function of time only. The solutions for Δ_1 and Δ_2 now become

$$\left.\begin{aligned}\Delta_1 &= \beta_1 \cos\theta + \beta_2 \sin\theta + \frac{\bar{\omega}}{2}\frac{\partial}{\partial t}(h_5 \sin 2\theta - h_6 \cos 2\theta + a_1)\,,\\ \Delta_2 &= -\beta_1 \sin\theta + \beta_2 \cos\theta + \bar{\omega}^2 \frac{\partial h_1}{\partial t} + \frac{\bar{\omega}}{2}\frac{\partial h_2}{\partial t} + p\bar{\omega}\,,\end{aligned}\right\} \quad (980)$$

where h_1 and h_2 are defined according to equation (966).

49. *The integrability relations for a two-dimensional stellar system having circular symmetry.*—In § 48 we have obtained the solutions for a, b, h, Δ_1, and Δ_2, and we have now to discuss the further restrictions imposed on these solutions by the compatibility relations (III_9). These relations in polar co-ordinates are

$$\left.\begin{aligned}a\frac{\partial \mathfrak{B}}{\partial \bar{\omega}} + \frac{h}{\bar{\omega}}\frac{\partial \mathfrak{B}}{\partial \theta} + \frac{\partial \Delta_1}{\partial t} &= -\frac{1}{2}\frac{\partial \chi}{\partial \bar{\omega}}\,,\\ h\frac{\partial \mathfrak{B}}{\partial \bar{\omega}} + \frac{b}{\bar{\omega}}\frac{\partial \mathfrak{B}}{\partial \theta} + \frac{\partial \Delta_2}{\partial t} &= -\frac{1}{2\bar{\omega}}\frac{\partial \chi}{\partial \theta}\,,\\ \Delta_1 \frac{\partial \mathfrak{B}}{\partial \bar{\omega}} + \frac{\Delta_2}{\bar{\omega}}\frac{\partial \mathfrak{B}}{\partial \theta} &= +\frac{1}{2}\frac{\partial \chi}{\partial t}\,.\end{aligned}\right\} \quad (\mathrm{III}_{11})$$

The general discussion of the foregoing relations (III_{11}) will not be attempted in this paper. However, there is one important special case for which the integrability conditions, which the equations (III_{11}) give rise to, can be handled without much difficulty. This is the case when the potential function $\mathfrak{B}$ is characterized by circular symmetry, i.e.,

$$\mathfrak{B} \equiv \mathfrak{B}(\bar{\omega}, t) . \tag{981}$$

Apart from its obvious physical importance, the mathematical interest in this case arises chiefly from the circumstance that under steady-state conditions the circular symmetry for $\mathfrak{B}$ can be ***proved*** (§ 8). It is therefore natural that we should begin our analysis of nonsteady states by inquiring into the freedom which is gained by making $\mathfrak{B}$ a function of time as well but retaining its circularly symmetrical character.

Assuming, then, that $\mathfrak{B}$ has circular symmetry (Eq. [981]), the compatibility relations (III_{11}) take the simpler forms

$$\left.\begin{aligned} a\,\frac{\partial \mathfrak{B}}{\partial \bar{\omega}} + \frac{\partial \Delta_1}{\partial t} &= -\frac{1}{2}\,\frac{\partial \chi}{\partial \bar{\omega}}\,, \\ h\bar{\omega}\,\frac{\partial \mathfrak{B}}{\partial \bar{\omega}} + \bar{\omega}\,\frac{\partial \Delta_2}{\partial t} &= -\frac{1}{2}\,\frac{\partial \chi}{\partial \theta}\,, \\ \Delta_1\,\frac{\partial \mathfrak{B}}{\partial \bar{\omega}} &= +\frac{1}{2}\,\frac{\partial \chi}{\partial t}\,. \end{aligned}\right\} \tag{982}$$

Equations (982) give rise to the following integrability conditions:

$$\frac{\partial a}{\partial \theta}\,\frac{\partial \mathfrak{B}}{\partial \bar{\omega}} + \frac{\partial^2 \Delta_1}{\partial t \partial \theta} = h\,\frac{\partial \mathfrak{B}}{\partial \bar{\omega}} + \frac{\partial \Delta_2}{\partial t} + \bar{\omega}\,\frac{\partial}{\partial \bar{\omega}}\left(h\,\frac{\partial \mathfrak{B}}{\partial \bar{\omega}} + \frac{\partial \Delta_2}{\partial t}\right), \tag{983}$$

$$\frac{\partial}{\partial \bar{\omega}}\left(\Delta_1\,\frac{\partial \mathfrak{B}}{\partial \bar{\omega}}\right) + \frac{\partial}{\partial t}\left(a\,\frac{\partial \mathfrak{B}}{\partial \bar{\omega}}\right) + \frac{\partial^2 \Delta_1}{\partial t^2} = 0\,, \tag{984}$$

and

$$\frac{\partial}{\partial \theta}\left(\Delta_1\,\frac{\partial \mathfrak{B}}{\partial \bar{\omega}}\right) + \bar{\omega}\,\frac{\partial}{\partial t}\left(h\,\frac{\partial \mathfrak{B}}{\partial \bar{\omega}}\right) + \bar{\omega}\,\frac{\partial^2 \Delta_2}{\partial t^2} = 0\,. \tag{985}$$

In the general discussion of the foregoing equations we shall explicitly exclude the case

$$\frac{\partial \mathfrak{V}}{\partial \bar{\omega}} = C(t)\bar{\omega}\ , \tag{986}$$

where C is a function of time only. The relation (986) corresponds to the case of a quasi-elastic field of force. It is found that this case allows a much greater freedom than when (986) is not satisfied. We shall therefore reserve this case of a quasi-elastic field of force for a separate treatment (§ 57); in the meantime we shall assume that (986) is not satisfied.

50. *The discussion of equation (983).*—If we substitute for a, b, and h according to (967) in equation (983), we readily obtain

$$\left.\begin{aligned}(h_1\bar{\omega} + h_2)\bar{\omega}\,\frac{\partial^2 \mathfrak{V}}{\partial \bar{\omega}^2} + (2h_1\bar{\omega} - h_2)\,\frac{\partial \mathfrak{V}}{\partial \bar{\omega}}&\\ + \frac{\partial}{\partial t}\left(\Delta_2 + \bar{\omega}\,\frac{\partial \Delta_2}{\partial \bar{\omega}} - \frac{\partial \Delta_1}{\partial \theta}\right) &= 0\ .\end{aligned}\right\} \tag{987}$$

Again, substituting for Δ_1 and Δ_2 from (980) in the foregoing equation, we find

$$(h_1\bar{\omega} + h_2)\bar{\omega}\,\frac{\partial^2 \mathfrak{V}}{\partial \bar{\omega}^2} + (2h_1\bar{\omega} - h_2)\,\frac{\partial \mathfrak{V}}{\partial \bar{\omega}} + 3\bar{\omega}^2\,\frac{\partial^2 h_1}{\partial t^2} + 2\bar{\omega}\,\frac{dp}{dt} = 0\ . \tag{988}$$

Dividing the foregoing equation throughout by $(h_1\bar{\omega} + h_2)$ and differentiating the resulting equation partially with respect to θ, we obtain, after some minor transformations,

$$\left.\begin{aligned}3(h_1'h_2 - h_1h_2')\,\frac{\partial \mathfrak{V}}{\partial \bar{\omega}} + 3[(h_1\bar{\omega} + h_2)\ddot{h}_1' - (h_1'\bar{\omega} + h_2')\ddot{h}_1]\bar{\omega}&\\ - 2(h_1'\bar{\omega} + h_2')\dot{p} &= 0\ ,\end{aligned}\right\} \tag{989}$$

where we have used dots and primes to denote differentiations with respect to t and θ, respectively.

We shall first show that p is a constant. Let us suppose that this is not the case. It is, first of all, seen that if $\dot{p} \neq 0$, then either h_1 or $h_2 = 0$. For, if not, we can divide equation (989) throughout by

$(h_1'h_2 - h_1h_2')$ and differentiate with respect to θ; we obtain in this manner

$$\left.\begin{aligned} 3\bar{\omega}\frac{\partial}{\partial\theta}\left\{\frac{(h_1\bar{\omega}+h_2)\ddot{h}_1' - (h_1'\bar{\omega}+h_2')\ddot{h}_1}{h_1'h_2 - h_1h_2'}\right\} & \\ - 2\frac{\partial}{\partial\theta}\left\{\frac{h'\bar{\omega}+h_2'}{h_1'h_2 - h_1h_2'}\right\}\dot{p} &= 0\,. \end{aligned}\right\} \tag{990}$$

Since h_1 and h_2 are two functions independent of $\bar{\omega}$, we can equate the coefficients of the different powers of $\bar{\omega}$ in the foregoing equation separately to zero. Thus the terms in (990), independent of $\bar{\omega}$, give

$$\frac{\partial}{\partial\theta}\left(\frac{h_2'}{h_1'h_2 - h_1h_2'}\right)\dot{p} = 0\,; \tag{991}$$

or, since $\dot{p} \neq 0$ by hypothesis, we should have

$$\frac{\partial}{\partial\theta}\left(\frac{h_2'}{h_1'h_2 - h_1h_2'}\right) = 0\,; \tag{992}$$

or, performing the required differentiation, we find

$$h_2(h_1h_2' - 4h_1'h_2) = 0\,. \tag{993}$$

Equation (993) implies that either h_1 or h_2 must vanish. We have thus shown that if $\dot{p} \neq 0$, h_1 or h_2 equals zero. In either of these two cases equation (989) reduces to

$$3[(h_1\bar{\omega} + h_2)\ddot{h}_1' - (h_1'\bar{\omega} + h_2')\ddot{h}_1]\bar{\omega} - 2(h_1'\bar{\omega} + h_2')\dot{p} = 0\,. \tag{994}$$

If $h_1 = 0$, equation (994) implies that $h_2'\dot{p} = 0$, or $h_2 = 0$, since by hypothesis $\dot{p} \neq 0$. But from (988) it follows that $h_1 = h_2 = 0$ implies that $\dot{p} = 0$, which contradicts our hypothesis. Similarly, $h_2 = 0$ is also seen to lead to the same contradiction. We have thus shown that

$$p = \text{constant}\,. \tag{995}$$

Returning to equation (989), we now have

$$(h_1'h_2 - h_1h_2')\frac{\partial\mathfrak{V}}{\partial\bar{\omega}} + [(h_1\bar{\omega} + h_2)\check{h}_1' - (h_1'\bar{\omega} + h_2')\check{h}_1]\bar{\omega} = 0\,. \tag{996}$$

Since $\mathfrak{V}$ is assumed to be independent of θ, the foregoing equation implies that

$$\frac{\partial}{\partial\theta}\left\{\frac{(h_1\bar{\omega} + h_2)\check{h}_1' - (h_1'\bar{\omega} + h_2')\check{h}_1}{h_1'h_2 - h_1h_2'}\right\} = 0\,. \tag{997}$$

Performing the required differentiation in (997), and after some further reductions, we find that

$$(h_1\bar{\omega} + h_2)(\check{h}_1h_1' - h_1\check{h}_1') = 0\,. \tag{998}$$

Hence,

$$\check{h}_1h_1' - h_1\check{h}_1' = 0 \qquad (h_1 \neq 0)\,, \tag{999}$$

or

$$\frac{\partial}{\partial\theta}\left(\frac{\check{h}_1}{h_1}\right) = 0\,. \tag{1000}$$

Thus, $\check{h}_1/h_1$ is shown to be independent of θ.

Returning to equation (988), we have (since $\dot{p} = 0$)

$$(h_1\bar{\omega} + h_2)\bar{\omega}\frac{\partial^2\mathfrak{V}}{\partial\bar{\omega}^2} + (2h_1\bar{\omega} - h_2)\frac{\partial\mathfrak{V}}{\partial\bar{\omega}} + 3\bar{\omega}^2\check{h}_1 = 0\,. \tag{1001}$$

Dividing the foregoing equation throughout by h_1 and differentiating with respect to θ, we have (on using Eq. [1000])

$$\bar{\omega}\frac{\partial}{\partial\theta}\left(\frac{h_2}{h_1}\right)\frac{\partial^2\mathfrak{V}}{\partial\bar{\omega}^2} - \frac{\partial}{\partial\theta}\left(\frac{h_2}{h_1}\right)\frac{\partial\mathfrak{V}}{\partial\bar{\omega}} = 0\,. \tag{1002}$$

Hence, if $h_2 \neq 0$, equation (1002) implies that

$$\bar{\omega}\frac{\partial^2\mathfrak{V}}{\partial\bar{\omega}^2} = \frac{\partial\mathfrak{V}}{\partial\bar{\omega}}\,, \tag{1003}$$

or

$$\frac{\partial \mathfrak{B}}{\partial \bar{\omega}} = C(t)\bar{\omega} \, , \tag{1004}$$

a case which we have explicitly excluded. Hence, $h_2 = 0$.

Collecting the results obtained so far, we have

$$\left.\begin{aligned} p &= \text{constant} \; ; \qquad h_2 = 0 \, , \\ \frac{\dot{h}_1}{h_1} &= \text{function of time only.} \end{aligned}\right\} \tag{1005}$$

Hence, if $h_1 \not\equiv 0$, we can re-write equation (1001) as

$$\bar{\omega}^2 \frac{\partial^2 \mathfrak{B}}{\partial \bar{\omega}^2} + 2\bar{\omega} \frac{\partial \mathfrak{B}}{\partial \bar{\omega}} + 3\bar{\omega}^2 \left(\frac{\dot{h}_1}{h_1}\right) = 0 \, . \tag{1006}$$

Equation (1006) is easily integrated. We find

$$\frac{\partial \mathfrak{B}}{\partial \bar{\omega}} = -\left(\frac{\dot{h}_1}{h_1}\right)\bar{\omega} + \frac{D(t)}{\bar{\omega}^2} \, , \tag{1007}$$

where $D(t)$ is an arbitrary function of time.

We have thus shown that equation (983) requires $\mathfrak{B}$ to be of a rather special kind. This restriction on $\mathfrak{B}$ (required by equation [1007]) can, however, be avoided by setting $h_1 \equiv 0$. We shall later return to this question.

51. *The discussion of equations (984) and (985).*—The discussion in the preceding section has shown that $h_2 = 0$. Hence, our solutions for a, b, h, Δ_1, and Δ_2 (Eqs. [965] and [980]) simplify to

$$\left.\begin{aligned} a &= a_1 \, , \\ b &= a_1 - 2(h_3 \sin\theta - h_4 \cos\theta)\bar{\omega} + b_{20}\bar{\omega}^2 \, , \\ h &= (h_3 \cos\theta + h_4 \sin\theta)\,\bar{\omega} = h_1\bar{\omega} \, , \end{aligned}\right\} \tag{1008}$$

and

$$\left.\begin{aligned} \Delta_1 &= \beta_1 \cos\theta + \beta_2 \sin\theta + \frac{\bar{\omega}}{2}\frac{da_1}{dt} \, , \\ \Delta_2 &= -\beta_1 \sin\theta + \beta_2 \cos\theta + \bar{\omega}^2 \frac{\partial h_1}{\partial t} + p\bar{\omega} \, , \end{aligned}\right\} \tag{1009}$$

where b_{20} and p are constants.

Substituting for a and Δ_1 from the foregoing equations in (984), we find after some reductions that

$$\Delta_1 \frac{\partial^2 \mathfrak{P}}{\partial \bar{\omega}^2} + \frac{3}{2} \frac{da_1}{dt} \frac{\partial \mathfrak{P}}{\partial \bar{\omega}} + a_1 \frac{\partial^2 \mathfrak{P}}{\partial \bar{\omega} \partial t} + \frac{\partial^2 \Delta_1}{\partial t^2} = 0 . \tag{1010}$$

Differentiating equation (1010) with respect to θ, we obtain

$$\frac{\partial \Delta_1}{\partial \theta} \frac{\partial^2 \mathfrak{P}}{\partial \bar{\omega}^2} + \frac{\partial^3 \Delta_1}{\partial t^2 \partial \theta} = 0 ; \tag{1011}$$

or, using the expression for Δ_1 given by (1009), we have

$$(-\beta_1 \sin \theta + \beta_2 \cos \theta) \frac{\partial^2 \mathfrak{P}}{\partial \bar{\omega}^2} = -\frac{\partial^2}{\partial t^2} (-\beta_1 \sin \theta + \beta_2 \cos \theta) . \tag{1012}$$

If β_1 and β_2 do not vanish identically, equation (1012) will lead to a relation of the form

$$\frac{\partial^2 \mathfrak{P}}{\partial \bar{\omega}^2} = C(t) , \tag{1013}$$

where $C(t)$ is some function of time; but (1013) is the case we have explicitly excluded. Hence,

$$\beta_1 = \beta_2 = 0 . \tag{1014}$$

Hence,

$$\Delta_1 = \frac{\bar{\omega}}{2} \frac{da_1}{dt} ; \qquad \Delta_2 = \bar{\omega}^2 \frac{\partial h_1}{\partial t} + p\bar{\omega} . \tag{1015}$$

Equation (1010) now takes the form

$$\frac{\bar{\omega}}{2} \frac{da_1}{dt} \frac{\partial^2 \mathfrak{P}}{\partial \bar{\omega}^2} + a_1 \frac{\partial^2 \mathfrak{P}}{\partial \bar{\omega} \partial t} + \frac{3}{2} \frac{da_1}{dt} \frac{\partial \mathfrak{P}}{\partial \bar{\omega}} + \frac{\bar{\omega}}{2} \frac{d^3 a_1}{dt^3} = 0 . \tag{1016}$$

We have encountered here for the first time a differential equation which is later seen to play a very fundamental role in the whole theory.

Postponing the discussion of the differential equation (1016) to

§§ 52 and 54, we shall now pass on to a consideration of the equation (985).

We have shown that Δ_1 is independent of θ, and, since $\mathfrak{V}$ is also assumed to be independent of θ, equation (985) now reduces to

$$\frac{\partial}{\partial t}\left(h\frac{\partial \mathfrak{V}}{\partial \bar{\omega}}+\frac{\partial \Delta_2}{\partial t}\right)=0\ ; \tag{1017}$$

or, substituting for h and Δ_2 according to equations (1008) and (1015), we have

$$\frac{\partial}{\partial t}\left(h_1\frac{\partial \mathfrak{V}}{\partial \bar{\omega}}+\bar{\omega}\ddot{h}_1\right)=0\ . \tag{1018}$$

Equation (1018) is equivalent to

$$h_1\frac{\partial \mathfrak{V}}{\partial \bar{\omega}}+\bar{\omega}\ddot{h}_1=\psi(\bar{\omega},\theta)\ , \tag{1019}$$

where ψ is independent of t. Equation (1019) can be written alternatively as

$$\frac{\partial \mathfrak{V}}{\partial \bar{\omega}}=-\left(\frac{\ddot{h}_1}{h_1}\right)\bar{\omega}+\frac{\psi(\bar{\omega},\theta)}{h_1}\ . \tag{1020}$$

Comparing the two equations (1007) and (1020), we see that the compatibility of these two equations requires

$$\frac{\psi(\bar{\omega},\theta)}{h_1}=\frac{D(t)}{\bar{\omega}^2}\ . \tag{1021}$$

Hence,

$$\frac{\partial}{\partial \bar{\omega}}(\bar{\omega}^2\psi)=\frac{\partial}{\partial \bar{\omega}}(h_1 D)=0\ , \tag{1022}$$

or

$$\bar{\omega}^2\psi=\psi_0(\theta)\ , \tag{1023}$$

where ψ_0 is a function of θ only. Equation (1021) now reduces to

$$\psi_0(\theta)=\{h_3(t)\cos\theta+h_4(t)\sin\theta\}D(t)\ , \tag{1024}$$

where we have substituted explicitly for h_1. From the foregoing equation we obtain

$$\cos\theta \frac{d}{dt}(h_3 D) + \sin\theta \frac{d}{dt}(h_4 D) = \frac{\partial}{\partial t}\psi_0(\theta) = 0 . \qquad (1025)$$

From equation (1025) we readily conclude that

$$\left.\begin{aligned} h_3 D = \text{constant} ; \qquad h_4 D = \text{constant} , \\ \frac{h_3}{h_4} = \text{constant} . \end{aligned}\right\} \qquad (1026)$$

Equations (1026) enable us to express h_3, h_4, and D in terms of a single arbitrary function of time. We can thus write

$$h_3 = h_{30}\phi ; \qquad h_4 = h_{40}\phi , \qquad (1027)$$

and

$$D = \frac{q_0}{\phi} , \qquad (1028)$$

where h_{30}, h_{40}, and q_0 are constants and ϕ is an arbitrary function of time. Equations (1024) and (1021) now become

$$\psi_0(\theta) = q_0(h_{30}\cos\theta + h_{40}\sin\theta) \qquad (1029)$$

and

$$\frac{\psi(\bar{\omega}, \theta)}{h_1} = \frac{D(t)}{\bar{\omega}^2} = \frac{q_0}{\phi}\frac{1}{\bar{\omega}^2} . \qquad (1030)$$

Furthermore, we note that under our present circumstances equation (1000) is identically satisfied; in fact, we have

$$\frac{\ddot{h}_1}{h_1} = \frac{\ddot{\phi}}{\phi} . \qquad (1030')$$

Finally, equations (1007) and (1020) are now seen to be equivalent, and we have

$$\frac{\partial \mathfrak{V}}{\partial \bar{\omega}} = -\frac{\ddot{\phi}}{\phi}\bar{\omega} + \frac{q_0}{\phi}\frac{1}{\bar{\omega}^2} . \qquad (1031)$$

The restriction on the form of $\mathfrak{B}$ which (1031) implies can be avoided if we set $h_1 \equiv 0$[31], in which case the equations (983) and (985) will be satisfied identically and the integrability conditions reduce to the single differential equation (1016) (cf. § 54). We shall now proceed to the discussion of this equation.

52. *The discussion of the equation (1016): A special case.*—We shall begin our discussion of this important equation by considering its implications for the special case which arises when $h_1 \not\equiv 0$. For, as we have shown in §§ 50 and 51, if $h_1 \not\equiv 0$, then the force function is required to be of the form

$$\frac{\partial \mathfrak{B}}{\partial \bar{\omega}} = C(t)\bar{\omega} + \frac{D(t)}{\bar{\omega}^2}, \tag{1032}$$

where C and D were further specified in terms of a single arbitrary function of time (cf. Eq. [1031]). We shall therefore begin our study of the differential equation (1016) when the force function has the form (1032).[32]

Substituting, then, (1032) in (1016), we find after some minor reductions that

$$\left(2\frac{da_1}{dt}C + a_1\frac{dC}{dt} + \frac{1}{2}\frac{d^3a_1}{dt^3}\right)\bar{\omega} + \left(\frac{1}{2}\frac{da_1}{dt}D + a_1\frac{dD}{dt}\right)\frac{1}{\bar{\omega}^2} = 0\,. \tag{1033}$$

Since a_1, C, and D are all functions of time only, the foregoing equation is equivalent to the two equations

$$\frac{1}{2}\frac{da_1}{dt}D + a_1\frac{dD}{dt} = 0 \tag{1034}$$

and

$$2\frac{da_1}{dt}C + a_1\frac{dC}{dt} + \frac{1}{2}\frac{d^3a_1}{dt^3} = 0\,. \tag{1035}$$

[31] In addition, of course, to the further conditions ($h_5 = h_6 = \beta_1 = \beta_2 = 0$), which we have already shown to be necessary.

[32] We shall *not*, however, require for the present, that C and D have the restricted forms found for them in § 51.

Equation (1034) admits of immediate integration. We have

$$a_1 D^2 = \text{constant} . \tag{1036}$$

Equation (1035) is again an important differential equation; as we shall see, this equation repeatedly occurs in the further developments of the theory. This equation admits of a first integral:

Multiplying equation (1035) throughout by a_1, we have

$$\frac{d}{dt}(a_1^2 C) + \tfrac{1}{2} a_1 \frac{d^3 a_1}{dt^3} = 0 . \tag{1037}$$

It is, however, readily verified that

$$a_1 \frac{d^3 a_1}{dt^3} = \frac{d}{dt}\left\{ a_1 \frac{d^2 a_1}{dt^2} - \frac{1}{2}\left(\frac{da_1}{dt}\right)^2 \right\} . \tag{1038}$$

Hence, combining equations (1037) and (1038), we have

$$\frac{d}{dt}\left\{ a_1^2 C + \frac{a_1}{2}\frac{d^2 a_1}{dt^2} - \frac{1}{4}\left(\frac{da_1}{dt}\right)^2 \right\} = 0 , \tag{1039}$$

or

$$a_1^2 C + \frac{1}{2}\left\{ a_1 \frac{d^2 a_1}{dt^2} - \frac{1}{2}\left(\frac{da_1}{dt}\right)^2 \right\} = q , \tag{1040}$$

where q is a constant of integration. This is the required first integral. We can express it more conveniently as follows:

It is easily seen that

$$\frac{d^2}{dt^2}\sqrt{a_1} = \frac{1}{2}\frac{d}{dt}\left(\frac{1}{\sqrt{a_1}}\frac{da_1}{dt}\right) = \tfrac{1}{2} a_1^{-(3/2)}\left\{ a_1 \frac{d^2 a_1}{dt^2} - \frac{1}{2}\left(\frac{da_1}{dt}\right)^2 \right\} . \tag{1041}$$

Equations (1040) and (1041) can be combined to give

$$a_1^2 C + a_1^{3/2}\frac{d^2}{dt^2}\sqrt{a_1} = q , \tag{1042}$$

or, alternatively,

$$C = \frac{q}{a_1^2} - \frac{1}{\sqrt{a_1}}\frac{d^2}{dt^2}\sqrt{a_1} . \tag{1043}$$

Let

$$a_1 = \phi^2 . \tag{1044}$$ [33]

In terms of ϕ, the equations (1036) and (1044) take the neater forms

$$C = \frac{q}{\phi^4} - \frac{\ddot{\phi}}{\phi}; \qquad D = \frac{q_0}{\phi}, \tag{1045}$$

where q_0 is an arbitrary constant. We have thus shown that, when the force function has the form (1032), the differential equation (1016) determines C and D uniquely (i.e., apart from the two arbitrary constants q and q_0) in terms of a_1 (or ϕ). We have

$$\frac{\partial \mathfrak{V}}{\partial \bar{\omega}} = \left(\frac{q}{\phi^4} - \frac{\ddot{\phi}}{\phi}\right)\bar{\omega} + \frac{q_0}{\phi}\frac{1}{\bar{\omega}^2}. \tag{1046}$$

53. *Further discussion of the case* $h_1 \not\equiv 0$.—In § 52 we have determined the nature of the functions C and D in the force function (1032) as required by the differential equation (1016). The result of this analysis is expressed in equation (1046). If we now compare this result with (1031), which was derived as a consequence of the equations (983) and (985) and on the assumption that $h_1 \not\equiv 0$, we see that we should choose

$$q = 0 \tag{1047}$$

in equation (1046). Further, it is clear that the single arbitrary function of time which our solution finally involves can be regarded as defined by the equation (1044).

We can summarize the results of our discussion so far in the following terms:

If

$$\frac{\partial \mathfrak{V}}{\partial \bar{\omega}} \not\equiv C(t)\bar{\omega}, \tag{1048}$$

[33] At this stage we regard the introduction of ϕ simply as the choice of a new variable instead of a_1. At a later stage, when we go over to the consideration of the case which arose in our discussion in § 51, the arbitrary function of time then introduced can also be taken to be defined by Eq. (1044).

where C(t) *is a function of time only, then the integrability conditions require that*

$$h_5 = h_6 = \beta_1 = \beta_2 = 0 \tag{1049}$$

and that

$$p = \text{constant} . \tag{1050}$$

If

$$h_3, h_4 \neq 0 , \tag{1051}$$

then

$$\frac{\partial \mathfrak{B}}{\partial \bar{\omega}} = -\frac{\ddot{\phi}}{\phi}\bar{\omega} + \frac{q_0}{\phi}\frac{1}{\bar{\omega}^2} , \tag{1052}$$

where q_0 *is an arbitrary constant and* ϕ *is an arbitrary function of time. Further, the appropriate solutions for* a, b, h, Δ_1, *and* Δ_2 *are given by*

$$\left.\begin{aligned} a &= \phi^2 , \\ b &= \phi^2 - 2(h_{30}\sin\theta - h_{40}\cos\theta)\phi\bar{\omega} + b_{20}\bar{\omega}^2 , \\ h &= (h_{30}\cos\theta + h_{40}\sin\theta)\phi\bar{\omega} , \end{aligned}\right\} \tag{1053}$$

and

$$\left.\begin{aligned} \Delta_1 &= \phi\dot{\phi}\bar{\omega} , \\ \Delta_2 &= (h_{30}\cos\theta + h_{40}\sin\theta)\dot{\phi}\bar{\omega}^2 + p\bar{\omega} , \end{aligned}\right\} \tag{1054}$$

where h_{30}, h_{40}, b_{20}, *and* p *are all arbitrary constants.*

Now, a force function of the form

$$\frac{\partial \mathfrak{B}}{\partial \bar{\omega}} = C\bar{\omega} + \frac{D}{\bar{\omega}^2} \tag{1055}$$

can be interpreted as due to a central mass and to a "flat" uniform spheroidal distribution of mass coextensive with the system. This form of the force function has been extensively used since Oort first introduced it as a basis for a dynamical interpretation for the rotational constants A and B.[34] It is very remarkable that precisely this

[34] *B.A.N.*, **4**, 79, 1927.

form of the force function should play such a fundamental role in the dynamics of stellar systems in nonsteady states. In view of this it may be permissible to use our present solution (i.e., Eqs. [1053] and [1054]) as a basis for interpreting certain features of stellar motions which have not so far been explained entirely satisfactorily. Thus, it is known that the vertex of star-streaming does not point exactly to the galactic center and that the deviation amounts to as much as 10°.[35] It is therefore of interest that a force function of the form (1052), under the circumstances of a nonsteady state, does permit of a deviation of the vertex of amount ϵ given by

$$\tan 2\epsilon = \frac{2(h_{30}\cos\theta + h_{40}\sin\theta)\phi\bar{\omega}}{2(h_{30}\sin\theta - h_{40}\cos\theta)\phi\bar{\omega} - b_{20}\bar{\omega}^2} \,. \tag{1056}$$

It should be pointed out in this connection that under *steady-state* conditions a deviation of the vertex is possible only for the two special cases (cf. § 9)

$$\frac{\partial\mathfrak{V}}{\partial\bar{\omega}} = C\bar{\omega} \,; \qquad \frac{\partial\mathfrak{V}}{\partial\bar{\omega}} = \frac{D}{\bar{\omega}^2} \,. \tag{1057}$$

But these two cases are really "too special," and accordingly Lindblad, Heckmann, Oort, and others have rejected the possibility of explaining the observed deviation of the vertex on the basis of the formulae valid for either of the two cases (1057).[36] It should therefore be admitted that an interpretation of the facts of observations on the basis of our present solution (Eqs. [1053] and [1054]) is probably not general enough—it should, however, be emphasized that the form of the force function which now occurs as the special case has often been used in the past as a simple working hypothesis.

As we have already indicated, the force function (1052) can be interpreted as that due to a central mass, M_0, and a uniform spheroidal distribution of matter, of total mass M_s. If the spheroidal dis-

[35] For a useful summary of the observational side of these questions see J. H. Oort, *M.N.*, **99**, 369, 1939.

[36] Lindblad has, however, argued that the formulae for the two special cases (1057) (derived under steady-state circumstances) may be expected to be valid for a limited region of space and interval of time. (Cf. *Stockholms observatoriums annaler*, **12**, No. 4, 31.)

tribution is further considered as "extremely flat,"[37] then the force function can be expressed as

$$\frac{\partial\mathfrak{B}}{\partial\bar{\omega}} = \frac{3\pi G M_s}{4R_1^3}\bar{\omega} + \frac{GM_0}{\bar{\omega}^2}\,, \tag{1058}$$

where R_1 denotes the major axis of the spheroid. Comparing the foregoing expression with the theoretical form (1052) for $\partial\mathfrak{B}/\partial\bar{\omega}$, we identify

$$\left.\begin{aligned} -\frac{\ddot{\phi}}{\phi} &= \frac{3\pi G}{4R_1^3}M_s\,, \\ \frac{q_0}{\phi} &= GM_0\,. \end{aligned}\right\} \tag{1059}$$

This physical interpretation of the two terms of the equation (1052) requires us to restrict the character of the function $\phi(t)$, which has been left arbitrary so far. Thus, for all values of t which have a physical meaning we should have

$$\left.\begin{aligned} \infty > -\frac{\ddot{\phi}}{\phi} &\geqslant 0\,, \\ \infty > \frac{q_0}{\phi} &\geqslant 0\,. \end{aligned}\right\} \tag{1060}$$

We can proceed somewhat further with the interpretation of the equations (1059). Since, according to these equations, M_s, M_0, and R_1 are all expected to be functions of time, we are led to consider the following situation:

The variations of M_s *and* M_0 *with time are such that the total mass of the system,* $M = M_s + M_0$, *remains constant.* In other words, we picture to ourselves a continual *interchange* of matter between the central nucleus and the uniform distribution outside. In practice, it would be physically the most significant to consider the case where the process is one of the *transfer* of matter from the center to the outside. Also, in considering such a process we should, of course, allow for the possibility of the density of the spheroidal distribution

[37] And it must be extremely flat for our present two-dimensional theory to be applicable.

of mass (or its equivalent R_1) to depend on and vary with time; here, again, we may expect the density to decrease (and R_1 to increase) with time.

Under the circumstances envisaged we are able to write down a differential equation for ϕ. For, according to equation (1059),

$$\frac{q_0}{\phi} - \frac{4R_1^3}{3\pi}\frac{\ddot{\phi}}{\phi} = G(M_s + M_0) = GM , \tag{1061}$$

where M denotes the total mass and is a constant. Equation (1061) can be written alternatively

$$\ddot{\phi} = -\frac{3\pi GM}{4R_1^3}\left(\phi - \frac{q_0}{GM}\right) . \tag{1062}$$

Let R_0 be the value of R_1 at $t = 0$ (say). Introduce the quantities λ and γ, defined by

$$\lambda^2 = \frac{3\pi GM}{4R_0^3} ; \qquad \gamma = \frac{q_0}{GM} . \tag{1063}$$

Now, λ^{-1} is of the dimensions of time, and it is found that numerically

$$\frac{1}{\lambda} = 9.68 \times 10^7 \left(\frac{R_0}{10{,}000 \text{ parsecs}}\right)^{3/2}\left(\frac{10^{11} \times \odot}{M}\right)^{1/2} \text{ years} . \tag{1064}$$

Further, let

$$\vartheta = \phi - \gamma ; \qquad \rho(t) = \left(\frac{R_0}{R_1}\right)^3 . \tag{1065}$$

In terms of these new variables equation (1062) takes the simple form

$$\ddot{\vartheta} = -\lambda^2\rho\vartheta . \tag{1066}$$

The expressions for M_s and M_0 given by (1059) now reduce to

$$M_s = \frac{\vartheta}{\vartheta + \gamma} M ; \qquad M_0 = \frac{\gamma}{\vartheta + \gamma} M . \tag{1067}$$

Equation (1066) is to be further supplemented by the boundary condition (cf. Eq. [1065])

$$\rho = 1 \quad \text{at} \quad t = 0 . \tag{1068}$$

Further, the conditions (1060) now become equivalent to

$$\vartheta \geqslant 0 , \quad t \geqslant 0 . \tag{1069}$$

From physical considerations we may restrict the function $\rho(t)$ to satisfy

$$\dot{\rho} \leqslant 0 , \quad t \geqslant 0 . \tag{1070}$$

Finally, we can use, as a further boundary condition, $M_s = 0$ at $t = 0$; in other words, we assume that at $t = 0$ all the mass is concentrated at the center. Then

$$\vartheta = 0 \quad \text{at} \quad t = 0 . \tag{1071}$$

Now, equation (1066) involves the two unknown functions of time, ϑ and ρ; and consequently we cannot determine either of them as a function of t. Equation (1066) is therefore to be regarded simply as a relation between ϑ and ρ. However, for a physically significant solution, $\vartheta \geqslant 0$ and $\dot{\rho} \leqslant 0$, for $t \geqslant 0$, and these conditions already limit ϑ and ρ quite considerably. Thus, R_1 *cannot be a constant;* for, if $R_1 =$ constant, then $\rho \equiv 1$, and equation (1066), together with the boundary condition (1071), implies that

$$\vartheta = A \sin \lambda t , \tag{1072}$$

where A is a constant. The solution (1072) makes M_s negative for $t > \pi\lambda^{-1}$; consequently, this solution must be rejected. On the other hand, it is possible to write down dependences of ρ on t which yield for ϑ physically significant solutions. Thus, let us assume that

$$\rho = \frac{1}{(1 + at)^2} , \tag{1073}$$

where α is a constant. Equation (1066) can be reduced to

$$\xi^2 \frac{d^2\vartheta}{d\xi^2} + \frac{\lambda^2}{\alpha^2}\vartheta = 0 , \tag{1074}$$

where

$$\xi = 1 + \alpha t . \tag{1075}$$

The general solution of the differential equation (1074) takes one of three forms, depending on whether

$$2\lambda < \alpha ; \qquad 2\lambda = \alpha ; \qquad 2\lambda > \alpha . \tag{1076}$$

We have

$$\vartheta = \xi^{1/2}(A\xi^{+\frac{1}{2}\sqrt{1-4(\lambda^2/\alpha^2)}} + B\xi^{-\frac{1}{2}\sqrt{1-4(\lambda^2/\alpha^2)}}) \qquad (2\lambda < \alpha) , \tag{1077}$$

$$\vartheta = \xi^{1/2}(A \log \xi + B) \qquad (2\lambda = \alpha) , \tag{1078}$$

and

$$\left.\begin{aligned} \vartheta = \xi^{1/2}\Bigg(A \sin\left[\frac{1}{2}\sqrt{4\frac{\lambda^2}{\alpha^2} - 1}\log\xi\right] \\ + B\cos\left[\frac{1}{2}\sqrt{4\frac{\lambda^2}{\alpha^2} - 1}\log\xi\right]\Bigg) \qquad (2\lambda > \alpha) , \end{aligned}\right\} \tag{1079}$$

where A and B are constants. If we now impose the boundary condition (1071) on ϑ, the appropriate solutions are

$$\vartheta = A\xi^{1/2}(\xi^{+\frac{1}{2}\sqrt{1-4(\lambda^2/\alpha^2)}} - \xi^{-\frac{1}{2}\sqrt{1-4(\lambda^2/\alpha^2)}}) \qquad (2\lambda < \alpha) , \tag{1080}$$

$$\vartheta = A\xi^{1/2}\log\xi \qquad (2\lambda = \alpha) , \tag{1081}$$

and

$$\vartheta = A\xi^{1/2}\sin\left[\frac{1}{2}\sqrt{4\frac{\lambda^2}{\alpha^2} - 1}\log\xi\right] \qquad (2\lambda > \alpha) , \tag{1082}$$

The solution (1082) permits M_s to take negative values after a finite interval of time and should consequently be rejected. Thus, for ρ of the form (1073), $\alpha \geqslant 2\lambda$.

Finally, according to (1067), the expressions for M_s and M_0 for the two cases $2\lambda < \alpha$ and $2\lambda = \alpha$ are

$$\left.\begin{aligned} M_s &= \frac{A\xi^{1/2}(\xi^{\frac{1}{2}\sqrt{1-4(\lambda^2/\alpha^2)}} - \xi^{-\frac{1}{2}\sqrt{1-4(\lambda^2/\alpha^2)}})}{A\xi^{1/2}(\xi^{\frac{1}{2}\sqrt{1-4(\lambda^2/\alpha^2)}} - \xi^{-\frac{1}{2}\sqrt{1-4(\lambda^2/\alpha^2)}}) + \gamma} M\,, \\ M_0 &= \frac{\gamma}{A\xi^{1/2}(\xi^{\frac{1}{2}\sqrt{1-4(\lambda^2/\alpha^2)}} - \xi^{-\frac{1}{2}\sqrt{1-4(\lambda^2/\alpha^2)}}) + \gamma} M \end{aligned}\right\} (2\lambda < \alpha)\,, \tag{1083}$$

and

$$\left.\begin{aligned} M_s &= \frac{A\xi^{1/2}\log\xi}{A\xi^{1/2}\log\xi + \gamma} M\,, \\ M_0 &= \frac{\gamma}{A\xi^{1/2}\log\xi + \gamma} M \end{aligned}\right\} (2\lambda = \alpha)\,. \tag{1084}$$

According to both of the foregoing solutions,

$$M_s \to M\,; \qquad M_0 \to 0\,, \tag{1085}$$

as $t \to \infty$; at the same time, $\rho \to 0$ and $R_1 \to \infty$.

In other words, we have shown that, *starting from an initial state in which all the mass is concentrated at the center, the stellar system tends steadily toward an infinitely diffuse state.* This result has, of course, been proved only for the case in which the dependence of ρ on t is of the form (1073) (with $\alpha \geqslant 2\lambda$). However, the conclusion reached seems to have an intrinsic interest of its own. In any case, the discussion of this case indicates that a further consideration of the equation (1066), supplemented by the conditions (1068)–(1071), may lead to results of some importance.

54. *Further discussion of the compatibility relations.*—The discussion of the equations (983)–(985) in §§ 50–52 has shown that in our solution for a, b, h, Δ_1, and Δ_2

$$h_5 = h_6 = \beta_1 = \beta_2 = 0\,. \tag{1086}$$[38]

Further, if h_3 and h_4 are not both identically zero, then we are led to a perfectly definite explicit form for the force function.[39] Conse-

[38] It should be remembered that in proving this we explicitly excluded the case of the quasi-elastic field of force.

[39] This case has already been discussed at length in § 53.

quently, more general forms for the force function become possible only if h_3 and h_4 are also zero. In that case our solutions for a, b, h, Δ_1, and Δ_2 reduce to

$$a = a_1 \; ; \qquad b = a_1 + b_{20}\bar{\omega}^2 \; ; \qquad h = 0 \, , \tag{1087}$$

and

$$\Delta_1 = \tfrac{1}{2}\bar{\omega}\frac{da_1}{dt} \; ; \qquad \Delta_2 = p\bar{\omega} \, , \tag{1088}$$

where b_{20} and p are constants and a_1 is a function of time. The integrability conditions (983)–(985) now reduce to the single equation (1016):

$$a_1 \frac{\partial^2 \mathfrak{V}}{\partial\bar{\omega}\partial t} + \frac{\bar{\omega}}{2}\frac{da_1}{dt}\frac{\partial^2 \mathfrak{V}}{\partial\bar{\omega}^2} + \frac{3}{2}\frac{da_1}{dt}\frac{\partial \mathfrak{V}}{\partial\bar{\omega}} + \frac{\bar{\omega}}{2}\frac{d^3 a_1}{dt^3} = 0 \, . \tag{1089}$$

In view of the importance of the foregoing equation, it is of interest to derive this equation directly from the compatibility relations (982) with a, b, h, Δ_1, and Δ_2, given according to the equations (1087) and (1088). Substituting, then, for a, b, h, Δ_1, and Δ_2 from (1087) in (982), we have

$$\left.\begin{aligned} a_1 \frac{\partial \mathfrak{V}}{\partial\bar{\omega}} + \frac{\bar{\omega}}{2}\frac{d^2 a_1}{dt^2} &= -\frac{1}{2}\frac{\partial\chi}{\partial\bar{\omega}} \, , \\ 0 &= -\frac{1}{2}\frac{\partial\chi}{\partial\theta} \, , \\ \frac{\bar{\omega}}{2}\frac{da_1}{dt}\frac{\partial \mathfrak{V}}{\partial\bar{\omega}} &= +\frac{1}{2}\frac{\partial\chi}{\partial t} \, . \end{aligned}\right\} \tag{1090}$$

Hence χ is independent of θ, and the single integrability condition which the equations (1090) give rise to is

$$\frac{\partial}{\partial t}\left(a_1 \frac{\partial \mathfrak{V}}{\partial\bar{\omega}} + \frac{\bar{\omega}}{2}\frac{d^2 a_1}{dt^2}\right) + \frac{\partial}{\partial\bar{\omega}}\left(\frac{\bar{\omega}}{2}\frac{da_1}{dt}\frac{\partial \mathfrak{V}}{\partial\bar{\omega}}\right) = 0 \, , \tag{1091}$$

which is readily seen to be equivalent to equation (1089). For a general discussion of this equation it is convenient to introduce a new variable τ defined by

$$\tau = \tfrac{1}{2}\bar{\omega}^2 \, . \tag{1092}$$

We then have

$$d\tau = \bar{\omega} d\bar{\omega} \,. \tag{1093}$$

In terms of this new variable, equation (1091) can be re-written as

$$\frac{\partial}{\partial t}\left(a_1 \frac{\partial \mathfrak{P}}{\partial \tau} + \frac{1}{2}\frac{d^2 a_1}{dt^2}\right) + \frac{\partial}{\partial \tau}\left(\tau \frac{da_1}{dt}\frac{\partial \mathfrak{P}}{\partial \tau}\right) = 0 \,, \tag{1094}$$

or explicitly

$$a_1 \frac{\partial^2 \mathfrak{P}}{\partial t \partial \tau} + \tau \frac{da_1}{dt}\frac{\partial^2 \mathfrak{P}}{\partial \tau^2} + 2\frac{da_1}{dt}\frac{\partial \mathfrak{P}}{\partial \tau} + \frac{1}{2}\frac{d^3 a_1}{dt^3} = 0 \,. \tag{1095}$$

We shall now proceed to a general discussion of the foregoing equation. Differentiating this equation partially with respect to τ, we obtain

$$a_1 \frac{\partial^3 \mathfrak{P}}{\partial t \partial \tau^2} + \tau \frac{da_1}{dt}\frac{\partial^3 \mathfrak{P}}{\partial \tau^3} + 3\frac{da_1}{dt}\frac{\partial^2 \mathfrak{P}}{\partial \tau^2} = 0 \,. \tag{1096}$$

We readily verify that the foregoing equation is equivalent to

$$a_1 \frac{\partial}{\partial t}\left(\tau^3 \frac{\partial^2 \mathfrak{P}}{\partial \tau^2}\right) + \tau \frac{da_1}{dt}\frac{\partial}{\partial \tau}\left(\tau^3 \frac{\partial^2 \mathfrak{P}}{\partial \tau^2}\right) = 0 \,. \tag{1097}$$

Equation (1097) is a homogeneous partial differential equation of the Lagrangian type. The subsidiary equation is

$$\frac{dt}{a_1} = \frac{d\tau}{\tau \dfrac{da_1}{dt}} \,; \tag{1098}$$

or, somewhat differently,

$$\frac{1}{a_1}\frac{da_1}{dt} dt = \frac{d\tau}{\tau} \,. \tag{1099}$$

The foregoing equation admits of immediate integration. We have

$$\frac{\tau}{a_1} = \text{constant} \,. \tag{1100}$$

Hence, the solution of the equation (1097) can be written as

$$\tau^3 \frac{\partial^2 \mathfrak{V}}{\partial \tau^2} = F_0 \left(\frac{\tau}{a_1} \right) , \tag{1101}$$

where F_0 is an arbitrary function of the argument specified. From (1101) we obtain

$$\frac{\partial \mathfrak{V}}{\partial \tau} = C(t) + \int^{\tau} \frac{1}{\tau^3} F_0 \left(\frac{\tau}{a_1} \right) d\tau , \tag{1102}$$

where $C(t)$ is a function of time only. We can express the result (1102) more conveniently as

$$\frac{\partial \mathfrak{V}}{\partial \tau} = C(t) + \frac{1}{a_1^2} F_1 \left(\frac{\tau}{a_1} \right) , \tag{1103}$$

where

$$F_1(\xi) = \int^{\xi} \frac{1}{\xi^3} F_0(\xi) d\xi ; \tag{1104}$$

consequently, we can regard F_1 in (1103) as an arbitrary function of the argument. Substituting for $\partial \mathfrak{V}/\partial \tau$ from (1103) in our original equation (1095), we have

$$\left. \begin{aligned} a_1 \left(\frac{dC}{dt} - \frac{2}{a_1^3} \frac{da_1}{dt} F_1 - \frac{\tau}{a_1^4} \frac{da_1}{dt} F_1' \right) + \frac{\tau}{a_1^3} \frac{da_1}{dt} F_1' \\ + 2 \frac{da_1}{dt} \left(C + \frac{1}{a_1^2} F_1 \right) + \frac{1}{2} \frac{d^3 a_1}{dt^3} = 0 , \end{aligned} \right\} \tag{1105}$$

where F_1' denotes the derivative of F_1 with respect to its argument τ/a_1. It is seen that in (1105) all the terms involving F_1 or F_1' cancel out, and we are left with

$$a_1 \frac{dC}{dt} + 2 \frac{da_1}{dt} C + \frac{1}{2} \frac{d^3 a_1}{dt^3} = 0 , \tag{1106}$$

an equation which we have already encountered once (Eq. [1035]). As before, equation (1106) leads to (cf. Eq. [1043])

$$C = \frac{q}{\phi^4} - \frac{\ddot{\phi}}{\phi} , \tag{1107}$$

where q is a constant and

$$a_1 = \phi^2 . \tag{1108}$$

Combining, now, equations (1103) and (1107), we have

$$\frac{\partial \mathfrak{B}}{\partial \tau} = \left(\frac{q}{\phi^4} - \frac{\ddot{\phi}}{\phi}\right) + \frac{1}{\phi^4} F_1\left(\frac{\tau}{\phi^2}\right) . \tag{1109}$$

Reverting to our original variable $\bar{\omega}$, we have

$$\frac{\partial \mathfrak{B}}{\partial \bar{\omega}} = \left(\frac{q}{\phi^4} - \frac{\ddot{\phi}}{\phi}\right) \bar{\omega} + \frac{1}{\phi^3} F_2\left(\frac{\bar{\omega}}{\phi}\right) , \tag{1110}$$

where

$$F_2(\xi) = \xi F_1(\xi) . \tag{1111}$$

In equation (1110) we can regard F_2 as an arbitrary function of the argument specified. Equation (1110) gives us explicitly the most general form for the force function which is consistent with the integrability conditions. The force function is seen to involve time only through the single arbitrary function $\phi(t)$. It is further seen that our present result for the force function is a generalization of the result obtained in § 52.

Finally, the solution for $\mathfrak{B}$ is obtained by integrating equation (1110). We have

$$\mathfrak{B} = \frac{1}{2}\left(\frac{q}{\phi^4} - \frac{\ddot{\phi}}{\phi}\right) \bar{\omega}^2 + \frac{1}{\phi^2} \mathfrak{B}_1\left(\frac{\bar{\omega}}{\phi}\right) + \mathfrak{B}_0(t) , \tag{1112}$$

where

$$\mathfrak{B}_1(\xi) = \int^{\xi} F_2(\xi) d\xi \tag{1113}$$

and where $\mathfrak{B}_0(t)$ is an arbitrary function of time only.[40]

We can summarize the results of our present discussion in the following terms:

[40] In all practical applications we need only the force function, and consequently the occurrence of the arbitrary function $\mathfrak{B}_0(t)$ in (1112) is of no interest and can be ignored.

The most general form for the force function consistent with the compatibility relations (982) is given by

$$\frac{\partial \mathfrak{V}}{\partial \bar{\omega}} = \left(\frac{q}{\phi^4} - \frac{\ddot{\phi}}{\phi}\right)\bar{\omega} + \frac{1}{\phi^3} F_2\left(\frac{\bar{\omega}}{\phi}\right), \tag{1114}$$

where ϕ *and* F_2 *are arbitrary functions of* t *and* $\bar{\omega}/\phi$, *respectively, and* q *is a constant. Further, the appropriate solutions for* a, b, h, Δ_1, *and* Δ_2 *are given by*

$$a = \phi^2 \; ; \qquad b = \phi^2 + b_{20}\bar{\omega}^2 \; ; \qquad h = 0 \, , \tag{1115}$$

and

$$\Delta_1 = \bar{\omega}\phi\dot{\phi} \; ; \qquad \Delta_2 = p\bar{\omega} \, , \tag{1116}$$

where b_{20} *and* p *are constants.*

From (1116) we readily obtain the expressions for the components (Π_0 and Θ_0) of the motion of the local centroid along the radial and the transverse directions, respectively. We have

$$\Pi_0 = \bar{\omega}\frac{\dot{\phi}}{\phi} \; ; \qquad \Theta_0 = \frac{p\bar{\omega}}{b_{20}\bar{\omega}^2 + \phi^2} \, . \tag{1117}$$

Thus, in addition to differential rotations, we also have motions in the radial directions specified by Π_0. This field of radial motions corresponds to an expansion or a contraction, depending on the sign of the rate of change of ϕ. If $\dot{\phi} > 0$, Π_0 corresponds to a general expansion proportional to the distance from the center; on the other hand, we have a similar field of contraction if $\dot{\phi} < 0$. Finally, $\Pi_0 \neq 0$ only when $\dot{\phi} \neq 0$; in other words, *the field of radial motions is directly connected with the nonsteady character of the system.* Physical considerations seem to suggest a general expansion rather than a contraction. This would imply that

$$\dot{\phi} \geqslant 0 \, . \tag{1118}$$

The interpretation of the foregoing result is of interest. Now, according to (1115) and our assumption concerning the distribution function Ψ (Eq. [928]), we have

$$\Psi = \Psi\{\phi^2(\Pi - \Pi_0)^2 + (\phi^2 + b_{20}\bar{\omega}^2)(\Theta - \Theta_0)^2 + \sigma\} \, . \tag{1119}$$

If we assume for $\Psi(x)$ a Gaussian form, then

$$\Psi = e^{-[\phi^2(\Pi-\Pi_0)^2+(\phi^2+b_{20}\bar{\omega}^2)(\Theta-\Theta_0)^2+\sigma]} \; . \tag{1120}$$

According to (1120), we have

$$\overline{|\Pi - \Pi_0|} = \frac{1}{\phi\sqrt{\pi}} \; ; \; \overline{|\Theta - \Theta_0|} = \frac{1}{\sqrt{(\phi^2 + b_{20}\bar{\omega}^2)\pi}} \; . \tag{1121}$$

Thus, $\dot{\phi} > 0$ implies that *the mean residual speed in the radial and in the transverse directions decreases with time.*

The terms in the expressions for the radial velocity (ρ) and the transverse velocity (T) due to the field of differential motions (1117) are readily obtained. We find that

$$\left.\begin{aligned} \rho &= r\left[\frac{\dot{\phi}}{\phi} + A \sin 2(l - l_0)\right] , \\ T &= r[B + A \cos 2(l - l_0)] , \end{aligned}\right\} \tag{1122}$$

where

$$A = \frac{1}{2}\left(\frac{\partial\Theta_0}{\partial\bar{\omega}} - \frac{\Theta_0}{\bar{\omega}}\right) ; \qquad B = \frac{1}{2}\left(\frac{\partial\Theta_0}{\partial\bar{\omega}} + \frac{\Theta_0}{\bar{\omega}}\right) \tag{1123}$$

and with the other symbols having their usual meanings. In the expression for the radial velocity we have a K-term proportional to the distance. It does not seem likely that we can allow such a K-term to be greater than (say) 1 kilometer per second per thousand parsecs. Thus,

$$\frac{\dot{\phi}}{\phi} \simeq 1 \text{ km per sec per 1000 parsecs} , \tag{1124}$$

or, alternatively,

$$\frac{d \log \phi}{dt} \simeq 10^{-9} \text{ per year} . \tag{1125}$$

Hence, a K-term of the order of 1 km per sec per 1000 parsecs indicates a decrease in the mean residual speed by about 1 part in 10^9 per year.

Finally, for Π_0 and Θ_0 defined according to (1117) the direction of motion of the local centroid at a given point deviates from the transverse Θ-direction by an angle ψ, where

$$\tan \psi = \frac{\Pi_0}{\Theta_0} = \frac{1}{\omega_0} \frac{d \log \phi}{dt} \tag{1126}$$

and where ω_0 is the angular velocity of rotation. In the neighborhood of the sun $\omega_0 \simeq 10^{-15}$ radians per second; and with the value (1125) for $\dot{\phi}/\phi$ we have

$$\tan \psi \simeq 0.03 ; \qquad \psi \simeq 2^\circ , \tag{1127}$$

an amount which appears to be beyond the range of our present observational resources.

55. *The case* $\mathfrak{V} \equiv \mathfrak{V}(\bar{\omega})$.—It is of interest to consider whether, within the scope of our present theory, it is possible to satisfy all the conditions of the problem with a time-independent, circularly symmetrical potential function and still have the coefficients of the velocity ellipse and the motions of the local centroids depend on time. Such a possibility has been considered by Heckmann[41] and Lindblad[42] in somewhat different connections.

From (1114) it follows that, to obtain the most general form of the force function which is independent of time, we should require

$$\frac{q}{\phi^4} - \frac{\ddot{\phi}}{\phi} = \text{constant} = q_1^2 \text{ (say)} ; \tag{1128}$$

and

$$\frac{1}{\phi^3} F_2 \left(\frac{\bar{\omega}}{\phi} \right) \equiv \text{function of } \bar{\omega} \text{ only} . \tag{1129}$$

From (1129) it readily follows that

$$F_2 \left(\frac{\bar{\omega}}{\phi} \right) = q_0 \frac{\phi^3}{\bar{\omega}^3} , \tag{1130}$$

[41] *Göttingen Veröffentlichungen*, No. 41, 1934; No. 43, 1935.

[42] *M.N.*, **97**, 16, 1936.

where q_0 is a constant. Thus, the most general form of the force function independent of time is

$$\frac{d\mathfrak{B}}{d\bar{\omega}} = q_1^2\bar{\omega} + \frac{q_0}{\bar{\omega}^3} ; \tag{1131}$$

in other words, the sum of a quasi-elastic field of force and an inverse cube law of force. Further equation (1128) can be solved for ϕ. This equation is verified to be equivalent to[43]

$$\frac{d^3 a_1}{dt^3} + 4q_1^2 \frac{da_1}{dt} = 0 , \tag{1132}$$

where

$$a_1 = \phi^2 . \tag{1133}$$

Equation (1132) is readily integrated. We have

$$a_1 = a_0 + a_{01} \sin 2q_1(t - t_0) , \tag{1134}$$

where a_0, a_{01}, and t_0 are constants of integration. Thus, the solutions for a, b, h, Δ_1, and Δ_2 for the case under condition are

$$\left.\begin{aligned} a &= a_0 + a_{01} \sin 2q_1(t - t_0) , \\ b &= a_0 + a_{01} \sin 2q_1(t - t_0) + b_{20}\bar{\omega}^2 , \\ h &= 0 , \end{aligned}\right\} \tag{1135}$$

and

$$\Delta_1 = a_{01}q_1\bar{\omega} \cos 2q_1(t - t_0) ; \qquad \Delta_2 = p\bar{\omega} . \tag{1136}$$

Thus, a and b vary periodically with time with a frequency q_1/π. Since a and b are essentially positive quantities, we should require

$$a_0 > a_{01} . \tag{1137}$$

We have thus shown that with a force function of the form (1131) we have a stellar system the kinematical parameters describing which have an oscillatory character.

[43] Equation (1128) is equivalent to the equation (1040) with $C = q_1^2 =$ constant. But we obtained equation (1040) as the first integral of (1035). Hence, equation (1128) is equivalent to equation (1035) with $C = a$ constant $= q_1^2$.

It is remarkable that the force function (1131) should play this critical role in the general theory. However, this is probably connected with Newton's theorem of revolving orbits[44] and with the circumstance that the equations of motion of a particle in the central field of force (1131) can be integrated out explicitly in terms of elementary functions.

The equation of the orbit of a particle of unit mass in the field of force (1131) can be written down in the form[45]

$$\theta = \int^{\bar{\omega}} \left\{ c_1 - \frac{2}{c_2^2} \int^{\bar{\omega}} \left(q_1^2 \bar{\omega} + \frac{q_0}{\bar{\omega}^3} \right) d\bar{\omega} - \frac{1}{\bar{\omega}^2} \right\}^{-1/2} \frac{d\bar{\omega}}{\bar{\omega}^2} , \qquad (1138)$$

where c_1 and c_2 are constants of integration.[46] After some minor reductions we obtain

$$\theta = -\frac{1}{2} \int \frac{d(u^2)}{\sqrt{\left(\frac{q_0}{c_2^2} - 1 \right) u^4 + c_1 u^2 - \left(\frac{q_1}{c_2} \right)^2}} , \qquad (1139)$$

where $u = 1/\bar{\omega}$. From (1139) we obtain

$$\left. \begin{aligned} 2k^2u^2 + c_1 &= \sqrt{c_1^2 + 4k^2 \left(\frac{q_1}{c_2} \right)^2} \cosh 2k(\theta + \epsilon) , \\ &\qquad \text{where } k^2 = \frac{q_0}{c_2^2} - 1 \text{ when } q_0 > c_2^2 , \\ -2k^2u^2 + c_1 &= \sqrt{c_1^2 - 4k^2 \left(\frac{q_1}{c_2} \right)^2} \cos 2k(\theta + \epsilon) , \\ &\qquad \text{where } k^2 = 1 - \frac{q_0}{c_2^2} \text{ when } q_0 < c_2^2 , \\ c_1 u^2 - \left(\frac{q_1}{c_2} \right)^2 &= c_1^2 (\theta + \epsilon)^2 , \qquad \text{when } q_0 = c_2^2 , \end{aligned} \right\} \qquad (1140)$$

where in each case ϵ is a constant of integration. The foregoing solutions for the orbit of a particle in the field of force (1131) have a certain resemblance with Cotes's spirals, which represent the possible

[44] Cf. E. T. Whittaker, *Analytical Dynamics*, p. 83, Cambridge, Eng., 1936.

[45] *Ibid.*, p. 80.

[46] c_2 is the "constant of areas."

orbits in a purely inverse cube field. It would be of interest to study further the equations of motion in the central field (1131) in order to find explicitly the first integrals and approach the problem of specifying the distribution function Ψ from the point of view of the Jeans-Poincaré theorem.

56. *The solution for* χ.—We shall now complete our discussion of the compatibility relations by obtaining explicitly the solutions for χ appropriate to the different cases.

(i) *The general case.*—As we have already shown in § 55, the most general form for the potential function consistent with the integrability conditions is given (cf. Eq. [1112]) by

$$\mathfrak{V} = \frac{1}{2}\left(\frac{q}{\phi^4} - \frac{\ddot{\phi}}{\phi}\right)\bar{\omega}^2 + \frac{1}{\phi^2}\mathfrak{V}_1\left(\frac{\bar{\omega}}{\phi}\right) + \mathfrak{V}_0(t) . \tag{1141}$$

We notice that

$$\frac{\partial \mathfrak{V}_1}{\partial \bar{\omega}} = \frac{1}{\phi}\mathfrak{V}_1' ; \qquad \frac{\partial \mathfrak{V}_1}{\partial t} = -\frac{\bar{\omega}}{\phi^2}\dot{\phi}\mathfrak{V}_1' , \tag{1142}$$

where $\mathfrak{V}_1'$ denotes the derivative of $\mathfrak{V}_1$ with respect to the argument $\bar{\omega}/\phi$. Further, the solutions for a, b, h, Δ_1, and Δ_2 appropriate to this case are (cf. Eqs. [1115] and [1116])

$$a = \phi^2 ; \qquad b = \phi^2 + b_{20}\bar{\omega}^2 ; \qquad h = 0 , \tag{1143}$$

and

$$\Delta_1 = \bar{\omega}\phi\dot{\phi} ; \qquad \Delta_2 = p\bar{\omega} . \tag{1144}$$

Substituting for $\mathfrak{V}$, a, b, h, Δ_1, and Δ_2 according to the foregoing relations in the equations (982), we obtain

$$-\frac{1}{2}\frac{\partial \chi}{\partial \bar{\omega}} = \phi^2\left[\left(\frac{q}{\phi^4} - \frac{\ddot{\phi}}{\phi}\right)\bar{\omega} + \frac{1}{\phi^3}\mathfrak{V}_1'\right] + \bar{\omega}([\dot{\phi}]^2 + \phi\ddot{\phi}) , \tag{1145}$$

$$-\frac{1}{2}\frac{\partial \chi}{\partial \theta} = 0 , \tag{1146}$$

$$+\frac{1}{2}\frac{\partial \chi}{\partial t} = \bar{\omega}\phi\dot{\phi}\left[\left(\frac{q}{\phi^4} - \frac{\ddot{\phi}}{\phi}\right)\bar{\omega} + \frac{1}{\phi^3}\mathfrak{V}_1'\right] . \tag{1147}$$

Equations (1145) and (1147) can be simplified further to take the forms

$$-\frac{1}{2}\frac{\partial\chi}{\partial\bar{\omega}} = \left(\frac{q}{\phi^2} + [\dot{\phi}]^2\right)\bar{\omega} + \frac{1}{\phi}\mathfrak{V}_1' , \tag{1148}$$

$$+\frac{1}{2}\frac{\partial\chi}{\partial t} = \left(\frac{q}{\phi^3}\dot{\phi} - \dot{\phi}\ddot{\phi}\right)\bar{\omega}^2 + \bar{\omega}\frac{\dot{\phi}}{\phi^2}\mathfrak{V}_1' ; \tag{1149}$$

on the other hand, equation (1146) simply states that χ is independent of θ. Using the relations (1142), we can re-write the foregoing equations as

$$-\frac{1}{2}\frac{\partial\chi}{\partial\bar{\omega}} = \left(\frac{q}{\phi^2} + [\dot{\phi}]^2\right)\bar{\omega} + \frac{\partial\mathfrak{V}_1}{\partial\bar{\omega}} , \tag{1150}$$

$$+\frac{1}{2}\frac{\partial\chi}{\partial t} = \left(\frac{q}{\phi^3}\dot{\phi} - \dot{\phi}\ddot{\phi}\right)\bar{\omega}^2 - \frac{\partial\mathfrak{V}_1}{\partial t} . \tag{1151}$$

From the equations (1150) and (1151) we readily find the solution for χ. We have

$$-\tfrac{1}{2}\chi = \frac{1}{2}\left(\frac{q}{\phi^2} + [\dot{\phi}]^2\right)\bar{\omega}^2 + \mathfrak{V}_1\left(\frac{\bar{\omega}}{\phi}\right) + \text{constant} . \tag{1152}$$

Since (cf. Eq. [935])

$$-\chi = Q_0 + \sigma , \tag{1153}$$

where Q_0 is defined according to equation (934), we have for the case under consideration

$$-\chi = a\Pi_0^2 + b\Theta_0^2 + \sigma ; \tag{1154}$$

or, substituting for Π_0 and Θ_0 according to (1117), we have

$$-\chi = \bar{\omega}^2[\dot{\phi}]^2 + \frac{p^2\bar{\omega}^2}{\phi^2 + b_{20}\bar{\omega}^2} + \sigma . \tag{1155}$$

Combining the relations (1152) and (1155), we find

$$\sigma = \left(\frac{q}{\phi^2} - \frac{p^2}{\phi^2 + b_{20}\bar{\omega}^2}\right)\bar{\omega}^2 + 2\mathfrak{V}_1\left(\frac{\bar{\omega}}{\phi}\right) ; \tag{1156}$$

we shall have occasion to use this relation later.

(ii) *The case* $h_1 \neq 0$.—As we have already seen, the force function assumes an explicit form for this case. We have (cf. Eq. [1052])

$$\frac{\partial \mathfrak{V}}{\partial \bar{\omega}} = -\frac{\ddot{\phi}}{\phi}\bar{\omega} + \frac{q_0}{\phi}\frac{1}{\bar{\omega}^2}\,. \tag{1157}$$

Further, the appropriate solutions for a, b, h, Δ_1, and Δ_2 are given in equations (1053) and (1054). Substituting these relations in the equations (982), we find

$$\left.\begin{aligned} -\frac{1}{2}\frac{\partial \chi}{\partial \bar{\omega}} &= \bar{\omega}[\dot{\phi}]^2 + \frac{q_0\phi}{\bar{\omega}^2}\,, \\ -\frac{1}{2}\frac{\partial \chi}{\partial \theta} &= q_0(h_{30}\cos\theta + h_{40}\sin\theta)\,, \\ +\frac{1}{2}\frac{\partial \chi}{\partial t} &= -\bar{\omega}^2\dot{\phi}\ddot{\phi} + \frac{q_0\dot{\phi}}{\bar{\omega}}\,. \end{aligned}\right\} \tag{1158}$$

It is readily verified that the appropriate solution for χ is given by

$$-\tfrac{1}{2}\chi = \tfrac{1}{2}[\dot{\phi}]^2\bar{\omega}^2 + q_0\left[(h_{30}\sin\theta - h_{40}\cos\theta) - \frac{\phi}{\bar{\omega}}\right]. \tag{1159}$$

(iii) *The case* $\mathfrak{V} \equiv \mathfrak{V}(\bar{\omega})$.—The solution for this case is obtained by setting

$$\mathfrak{V}_1 = -\frac{1}{2}\frac{q_0\phi^2}{\bar{\omega}^2} \tag{1160}$$

and (cf. Eqs. [1133] and [1134])

$$\phi^2 = a_0 + a_{01}\sin 2q_1(t - t_0) \tag{1161}$$

in equation (1152).

57. *The case of a quasi-elastic field of force.*—To complete our present discussion of the two-dimensional problem, we should finally consider the case of the quasi-elastic field of force which we have so far explicitly excluded. For this case we can write

$$\frac{\partial \mathfrak{V}}{\partial \bar{\omega}} = C(t)\bar{\omega}\,, \tag{1162}$$

where $C(t)$ is a function of time only. We have now to discuss the integrability conditions (983)–(985) when the force function assumes the form (1162).

Now, equation (983) has been shown to reduce to (988). Substituting for $\partial\mathfrak{V}/\partial\bar{\omega}$ from (1162) in this equation, we obtain

$$3\bar{\omega}^2\left(\frac{\partial^2 h_1}{\partial t^2}+Ch_1\right)+2\bar{\omega}\frac{dp}{dt}=0\,. \tag{1163}$$

Hence,

$$\frac{\partial^2 h_1}{\partial t^2}=-Ch_1\,;\qquad p=\text{constant}\,. \tag{1164}$$

Using the explicit form for h_1 (Eq. [966]), we find that the first of the foregoing equations splits into two differential equations:

$$\frac{d^2 h_3}{dt^2}=-Ch_3\,;\qquad \frac{d^2 h_4}{dt^2}=-Ch_4\,. \tag{1165}$$

Considering, next, equation (984), we find that for the force function (1162) this equation reduces to

$$\left.\begin{aligned}C(\beta_1\cos\theta+\beta_2\sin\theta)+\frac{\partial^2}{\partial t^2}(\beta_1\cos\theta+\beta_2\sin\theta)\\+\bar{\omega}\left(2C\frac{\partial a}{\partial t}+a\frac{dC}{dt}+\frac{1}{2}\frac{\partial^3 a}{\partial t^3}\right)=0\,.\end{aligned}\right\} \tag{1166}$$

Since a is independent of $\bar{\omega}$, the foregoing equation is seen to be equivalent to the three equations

$$\frac{d^2\beta_1}{dt^2}=-C\beta_1\,;\qquad \frac{d^2\beta_2}{dt^2}=-C\beta_2\,, \tag{1167}$$

and

$$2C\frac{\partial a}{\partial t}+a\frac{dC}{dt}+\frac{1}{2}\frac{\partial^3 a}{\partial t^3}=0\,. \tag{1168}$$

Substituting for a according to (965) in equation (1168), we find that a_1, h_5, and h_6 all satisfy the differential equation

$$2C\frac{d\chi_1}{dt}+\chi_1\frac{dC}{dt}+\frac{1}{2}\frac{d^3\chi_1}{dt^3}=0\qquad(\chi_1=a_1,\,h_5,\text{ or }h_6)\,, \tag{1169}$$

where χ_1 stands for any one of the quantities a_1, h_5, or h_6. Equation (1169) is of the same form as (1035). We can therefore write (cf. Eqs. [1043] and [1044])

$$C = \frac{q}{\chi_1^2} - \frac{1}{\sqrt{\chi_1}} \frac{d^2}{dt^2}\sqrt{\chi_1} \qquad (\chi_1 = a_1, h_5, \text{ or } h_6) \; , \tag{1170}$$

where q is an arbitrary constant. If in equation (1170) we regard C as a known function of t, then a_1, h_5, and h_6 are determined in terms of $C(t)$ as the solutions of this differential equation. Further, according to equations (1165) and (1167), h_3, h_4, β_1, and β_2 all satisfy the differential equation

$$\frac{d^2\chi_2}{dt^2} = -C\chi_2 \qquad (\chi_2 = h_3, h_4, \beta_1, \text{ or } \beta_2) \; , \tag{1171}$$

where χ_2 stands for any of the quantities h_3, h_4, β_1, or β_2. Thus, equation (1171) determines h_3, h_4, β_1, and β_2 again in terms of $C(t)$.

Finally, considering equation (985), we find that this equation is identically true in virtue of the differential equations which a_1, etc., have been found to satisfy.

We shall discuss the implications of the equations (1170) and (1171) in Part XII (§§ 62 and 63), where the corresponding three-dimensional problem is considered. However, we may notice, meantime, the explicit form of the solutions for a_1, etc., for the case when C is a constant. In that case, let

$$C = q_1^2 = \text{constant} \; . \tag{1172}$$

According to the equations (1169) and (1171), we now have

$$\left.\begin{aligned} a_1 &= a_{10} + a_{11} \sin 2q_1(t + t_1) \; , \\ h_5 &= h_{50} + h_{51} \sin 2q_1(t + t_2) \; , \\ h_6 &= h_{60} + h_{61} \sin 2q_1(t + t_3) \; , \end{aligned}\right\} \tag{1173}$$

and

$$\left.\begin{aligned} h_3 &= h_{30} \sin q_1(t + t_4) \; , \\ h_4 &= h_{40} \sin q_1(t + t_5) \; , \\ \beta_1 &= \beta_{10} \sin q_1(t + t_6) \; , \\ \beta_2 &= \beta_{20} \sin q_1(t + t_7) \; . \end{aligned}\right\} \tag{1174}$$

In equations (1173) and (1174) a_{10}, a_{11}, h_{50}, h_{51}, h_{60}, h_{61}, h_{30}, h_{40}, β_{10}, β_{20}, t_1, ... , t_7 are all constants of integration—these, together with b_{20} and p, are the nineteen constants of integration which the general solution for the case (1172) involves.

This completes our discussion of the two-dimensional problem with a circularly symmetrical potential function.

XI. EXAMPLES OF THREE-DIMENSIONAL STELLAR SYSTEMS IN NONSTEADY STATES; SYSTEMS WITH SPHERICAL AND AXIAL SYMMETRIES

58. *Systems characterized by a spherically symmetrical potential function.*—Let us choose a fundamental Cartesian frame of reference with the origin of the system of co-ordinates at the center of symmetry of $\mathfrak{V}$. We can then write

$$\mathfrak{V}(x, y, z; t) \equiv \mathfrak{V}(\tau, t) , \tag{1175}$$

where

$$\tau = \tfrac{1}{2}(x^2 + y^2 + z^2) . \tag{1176}$$

According to equation (1175), we have

$$\text{grad } \mathfrak{V} = \boldsymbol{r} \frac{\partial \mathfrak{V}}{\partial \tau} , \tag{1177}$$

where $\boldsymbol{r}$ stands for the position vector (x, y, z). Thus, when $\mathfrak{V}$ is of the form (1175), the compatibility relations (III_{10}) can be rewritten in the forms

$$A\boldsymbol{r} \frac{\partial \mathfrak{V}}{\partial \tau} + \frac{\partial \boldsymbol{\Delta}}{\partial t} = -\tfrac{1}{2} \text{ grad } \chi \tag{1178}$$

and

$$\boldsymbol{\Delta} \cdot \boldsymbol{r} \frac{\partial \mathfrak{V}}{\partial \tau} = \frac{1}{2} \frac{\partial \chi}{\partial t} . \tag{1179}$$

The conditions for the integrability of the foregoing equations are readily obtained. We have (cf. Eqs. [958] and [960])

$$\text{curl}\left(A\boldsymbol{r} \frac{\partial \mathfrak{V}}{\partial \tau}\right) + \frac{\partial}{\partial t} (\text{curl } \boldsymbol{\Delta}) = 0 \tag{1180}$$

and

$$\operatorname{grad}\left(\boldsymbol{\Delta}\cdot r\,\frac{\partial\mathfrak{B}}{\partial\tau}\right)+\frac{\partial}{\partial t}\left(Ar\,\frac{\partial\mathfrak{B}}{\partial\tau}\right)+\frac{\partial^2\boldsymbol{\Delta}}{\partial t^2}=0\,. \tag{1181}$$

Our central problem now is the enumeration of the various circumstances under which the six simultaneous partial differential equations for $\mathfrak{B}$ which the equations (1180) and (1181) represent admit of a common solution of the form (1175).

As in Part X, we shall begin by explicitly excluding the case of a quasi-elastic field of force.[47] When this is done, it is seen that the solution for the coefficients of the velocity ellipsoid are formally the same as those that occur in the steady-state problem for the case of an inverse-square law of force.[48] Thus, according to our results in Part VIII, § 38, we have (cf. Eqs. [923] and [924])

$$\left.\begin{aligned} &h_{20}=f_{30}\,; \qquad f_{20}=g_{30}\,; \qquad g_{20}=h_{30}\,,\\ &f_{10}=g_{10}=h_{10}=f_{11}=g_{11}=h_{11}=0\,,\\ &a_0=b_0=c_0=-\kappa \quad (\text{say})\,. \end{aligned}\right\} \tag{1182}$$

The general solution for the coefficients of the velocity ellipsoid given in equation (941) now simplifies to

$$\left.\begin{aligned} a&=\kappa-2(h_{20}+h_{21}z)y-(2f_{20}+g_{40}z)z-h_{40}y^2\,,\\ b&=\kappa-2(f_{20}+f_{21}x)z-(2g_{20}+h_{40}x)x-f_{40}z^2\,,\\ c&=\kappa-2(g_{20}+g_{21}y)x-(2h_{20}+f_{40}y)y-g_{40}x^2\,,\\ f&=-h_{21}x^2+(f_{20}+f_{21}x)y+(h_{20}+g_{21}x)z+f_{40}yz\,,\\ g&=-f_{21}y^2+(g_{20}+g_{21}y)z+(f_{20}+h_{21}y)x+g_{40}zx\,,\\ h&=-g_{21}z^2+(h_{20}+h_{21}z)x+(g_{20}+f_{21}z)y+h_{40}xy\,, \end{aligned}\right\} \tag{1183}$$

where, according to equation (954),

$$f_{21},\ g_{21},\ h_{21},\ f_{40},\ g_{40},\ h_{40} \quad \text{are all constants}\,. \tag{1184}$$

[47] This case is considered in Part XII, § 64.

[48] This corresponds to the result $h_2=0$ in the two-dimensional analysis (§ 50, Eq. [1005]).

The corresponding solutions for the Δ's can now be written down. According to equations (1182) and (953), we have

$$\left.\begin{aligned}\Delta_1 &= y\left(y\frac{dg_{20}}{dt} - x\frac{dh_{20}}{dt}\right) - z\left(x\frac{df_{20}}{dt} - z\frac{dg_{20}}{dt}\right)\\ &\qquad + \frac{1}{2}\frac{d\kappa}{dt}x + \beta_3 y - \beta_2 z + \delta_1\,,\\ \Delta_2 &= z\left(z\frac{dh_{20}}{dt} - y\frac{df_{20}}{dt}\right) - x\left(y\frac{dg_{20}}{dt} - x\frac{dh_{20}}{dt}\right)\\ &\qquad + \frac{1}{2}\frac{d\kappa}{dt}y + \beta_1 z - \beta_3 x + \delta_2\,,\\ \Delta_3 &= x\left(x\frac{df_{20}}{dt} - z\frac{dg_{20}}{dt}\right) - y\left(z\frac{dh_{20}}{dt} - y\frac{df_{20}}{dt}\right)\\ &\qquad + \frac{1}{2}\frac{d\kappa}{dt}z + \beta_2 x - \beta_1 y + \delta_3\,.\end{aligned}\right\} \qquad (1185)$$

It is found that we are able to continue our discussion of the equations (1180) and (1181) in vector notation if we introduce the following vectors:

$$\boldsymbol{\beta} = (\beta_1, \beta_2, \beta_3)\,; \qquad \boldsymbol{\gamma} = (g_{20}, h_{20}, f_{20})\,; \qquad \boldsymbol{\delta} = (\delta_1, \delta_2, \delta_3)\,, \quad (1186)$$

and

$$\boldsymbol{\rho} = \boldsymbol{\gamma} \times \boldsymbol{r}\,. \qquad (1187)$$

In terms of these vectors the solution (1185) for the Δ's can be written as the single vector equation

$$\boldsymbol{\Delta} = \frac{\partial}{\partial t}(\boldsymbol{r} \times \boldsymbol{\rho}) + \frac{1}{2}\frac{d\kappa}{dt}\boldsymbol{r} + \boldsymbol{r} \times \boldsymbol{\beta} + \boldsymbol{\delta}\,. \qquad (1188)$$

Further, we readily verify that

$$\left.\begin{aligned}A\boldsymbol{r} &= \kappa\boldsymbol{r} + \boldsymbol{r} \times \boldsymbol{\rho}\,,\\ \boldsymbol{\Delta} \cdot \boldsymbol{r} &= \frac{1}{2}\frac{d\kappa}{dt}r^2 + \boldsymbol{r} \cdot \boldsymbol{\delta}\,.\end{aligned}\right\} \qquad (1189)$$

Equations (1178) and (1179) now take the forms

$$(\kappa \boldsymbol{r} + \boldsymbol{r} \times \boldsymbol{\rho}) \frac{\partial \mathfrak{P}}{\partial \tau} + \frac{\partial \boldsymbol{\Delta}}{\partial t} = -\tfrac{1}{2} \operatorname{grad} \chi \tag{1190}$$

and

$$\left(\tau \frac{d\kappa}{dt} + \boldsymbol{r} \cdot \boldsymbol{\delta}\right) \frac{\partial \mathfrak{P}}{\partial \tau} = +\frac{1}{2} \frac{\partial \chi}{\partial t}. \tag{1191}$$

We shall first consider the integrability condition (1180). We verify that

$$\left.\begin{aligned} \operatorname{curl}\left(A\boldsymbol{r} \frac{\partial \mathfrak{P}}{\partial \tau}\right) &= -2\tau \boldsymbol{\rho} \frac{\partial^2 \mathfrak{P}}{\partial \tau^2} - 3\boldsymbol{\rho} \frac{\partial \mathfrak{P}}{\partial \tau}, \\ \operatorname{curl} \boldsymbol{\Delta} &= -3 \frac{\partial \boldsymbol{\rho}}{\partial t} - 2\boldsymbol{\beta}. \end{aligned}\right\} \tag{1192}$$

Hence, equation (1180) reduces to

$$2\tau \boldsymbol{\rho} \frac{\partial^2 \mathfrak{P}}{\partial \tau^2} + 3\boldsymbol{\rho} \frac{\partial \mathfrak{P}}{\partial \tau} + 3 \frac{\partial^2 \boldsymbol{\rho}}{\partial t^2} + 2 \frac{d\boldsymbol{\beta}}{dt} = 0. \tag{1193}$$

Multiply the foregoing equation scalarly with $\boldsymbol{r}$. Since

$$\boldsymbol{r} \cdot \boldsymbol{\rho} = \boldsymbol{r} \cdot \boldsymbol{\gamma} \times \boldsymbol{r} = 0, \tag{1194}$$

we obtain

$$\boldsymbol{r} \cdot \frac{d\boldsymbol{\beta}}{dt} = 0. \tag{1195}$$

Thus $\boldsymbol{\beta}$ is a constant vector; in other words,

$$\beta_1, \beta_2, \beta_3 \quad \text{are all constants}. \tag{1196}$$

Equation (1193) thus reduces to

$$2\tau \boldsymbol{\rho} \frac{\partial^2 \mathfrak{P}}{\partial \tau^2} + 3\boldsymbol{\rho} \frac{\partial \mathfrak{P}}{\partial \tau} + 3 \frac{\partial^2 \boldsymbol{\rho}}{\partial t^2} = 0; \tag{1197}$$

or, substituting for $\boldsymbol{\rho}$ according to (1187), we have

$$\boldsymbol{r} \times \left(2\tau\boldsymbol{\gamma} \frac{\partial^2 \mathfrak{B}}{\partial \tau^2} + 3\boldsymbol{\gamma} \frac{\partial \mathfrak{B}}{\partial \tau} + 3 \frac{d^2 \boldsymbol{\gamma}}{dt^2} \right) = 0 . \tag{1198}$$

Since $\boldsymbol{\gamma}$ is a vector independent of the spatial co-ordinates, equation (1198) implies that

$$2\tau\boldsymbol{\gamma} \frac{\partial^2 \mathfrak{B}}{\partial \tau^2} + 3\boldsymbol{\gamma} \frac{\partial \mathfrak{B}}{\partial \tau} + 3 \frac{d^2 \boldsymbol{\gamma}}{dt^2} = 0 . \tag{1198$'$}$$

Equation (1198′) can be re-written as

$$\boldsymbol{\gamma} \frac{\partial}{\partial \tau} \left(\tau^{3/2} \frac{\partial \mathfrak{B}}{\partial \tau} \right) + \frac{3}{2} \tau^{1/2} \frac{d^2 \boldsymbol{\gamma}}{dt^2} = 0 , \tag{1199}$$

which can be integrated to give

$$\boldsymbol{\gamma} \frac{\partial \mathfrak{B}}{\partial \tau} + \frac{d^2 \boldsymbol{\gamma}}{dt^2} = \frac{\boldsymbol{\gamma}_0}{\tau^{3/2}} , \tag{1200}$$

where $\boldsymbol{\gamma}_0 = (g_0, h_0, f_0)$ is a vector depending on the time only. In order that the three equations for $\mathfrak{B}$, which the vector equation (1200) represents, be consistent, it is clearly necessary that

$$\frac{1}{g_{20}} \frac{d^2 g_{20}}{dt^2} = \frac{1}{h_{20}} \frac{d^2 h_{20}}{dt^2} = \frac{1}{f_{20}} \frac{d^2 f_{20}}{dt^2} \tag{1200$'$}$$

and

$$\frac{g_0}{g_{20}} = \frac{h_0}{h_{20}} = \frac{f_0}{f_{20}} = D(t) \text{ (say)} , \tag{1200$''$}$$

where $D(t)$ is an arbitrary function of time. Equation (1200) can now be expressed alternatively as

$$\frac{\partial \mathfrak{B}}{\partial \tau} = -\frac{1}{|\boldsymbol{\gamma}|^2} \boldsymbol{\gamma} \cdot \frac{d^2 \boldsymbol{\gamma}}{dt^2} + \frac{D(t)}{\tau^{3/2}} . \tag{1201}$$

The restriction on the form of $\mathfrak{B}$ which (1200) or (1201) requires can be avoided if and only if $\boldsymbol{\gamma} = 0$.

Consider next the second integrability equation (1181). Using the equations (1188), (1189), and (1196), we find

$$
\left.
\begin{aligned}
&\frac{\partial}{\partial t}\left(A\boldsymbol{r}\,\frac{\partial\mathfrak{B}}{\partial\tau}\right)=\frac{\partial}{\partial t}\left[(\kappa\boldsymbol{r}+\boldsymbol{r}\times\boldsymbol{\rho})\,\frac{\partial\mathfrak{B}}{\partial\tau}\right],\\
&\operatorname{grad}\left(\boldsymbol{\Delta}\cdot\boldsymbol{r}\,\frac{\partial\mathfrak{B}}{\partial\tau}\right)=\operatorname{grad}\left[\left(\tau\,\frac{d\kappa}{dt}+\boldsymbol{r}\cdot\boldsymbol{\delta}\right)\frac{\partial\mathfrak{B}}{\partial\tau}\right]\\
&\qquad=\boldsymbol{r}\left(\tau\,\frac{d\kappa}{dt}+\boldsymbol{r}\cdot\boldsymbol{\delta}\right)\frac{\partial^2\mathfrak{B}}{d\tau^2}+\left(\boldsymbol{r}\,\frac{d\kappa}{dt}+\boldsymbol{\delta}\right)\frac{\partial\mathfrak{B}}{\partial\tau},\\
&\frac{\partial^2\boldsymbol{\Delta}}{\partial t^2}=\boldsymbol{r}\times\frac{\partial^3\boldsymbol{\rho}}{\partial t^3}+\frac{1}{2}\,\boldsymbol{r}\,\frac{d^3\kappa_1}{dt^3}+\frac{d^2\boldsymbol{\delta}}{dt^2}\,.
\end{aligned}
\right\}
\qquad (1202)
$$

Hence, equation (1181) reduces to

$$
\left.
\begin{aligned}
&\boldsymbol{r}\left\{\frac{\partial}{\partial t}\left(\kappa\,\frac{\partial\mathfrak{B}}{\partial\tau}\right)+\tau\,\frac{d\kappa}{dt}\,\frac{\partial^2\mathfrak{B}}{\partial\tau^2}+\frac{d\kappa}{dt}\,\frac{\partial\mathfrak{B}}{\partial\tau}+\frac{1}{2}\,\frac{d^3\kappa}{dt^3}\right\}\\
&\quad+\boldsymbol{r}\times\frac{\partial}{\partial t}\left(\boldsymbol{\rho}\,\frac{\partial\mathfrak{B}}{\partial\tau}+\frac{\partial^2\boldsymbol{\rho}}{\partial t^2}\right)+\boldsymbol{r}(\boldsymbol{r}\cdot\boldsymbol{\delta})\,\frac{\partial^2\mathfrak{B}}{\partial\tau^2}+\boldsymbol{\delta}\,\frac{\partial\mathfrak{B}}{\partial\tau}+\frac{d^2\boldsymbol{\delta}}{dt^2}=0\,.
\end{aligned}
\right\}
\qquad (1203)
$$

Multiply the foregoing equation vectorially by $\boldsymbol{r}$. We get

$$
\boldsymbol{r}\times\left[\boldsymbol{r}\times\frac{\partial}{\partial t}\left(\boldsymbol{\rho}\,\frac{\partial\mathfrak{B}}{\partial\tau}+\frac{\partial^2\boldsymbol{\rho}}{\partial t^2}\right)\right]+\boldsymbol{r}\times\left(\boldsymbol{\delta}\,\frac{\partial\mathfrak{B}}{\partial\tau}+\frac{d^2\boldsymbol{\delta}}{dt^2}\right)=0\,. \qquad (1204)
$$

The first term in equation (1204) can be simplified by using the formula for the vector triple product. Thus,

$$
\left.
\begin{aligned}
&\boldsymbol{r}\times\left[\boldsymbol{r}\times\frac{\partial}{\partial t}\left(\boldsymbol{\rho}\,\frac{\partial\mathfrak{B}}{\partial\tau}+\frac{\partial^2\boldsymbol{\rho}}{\partial t^2}\right)\right]\\
&\qquad=\left[\boldsymbol{r}\cdot\frac{\partial}{\partial t}\left(\boldsymbol{\rho}\,\frac{\partial\mathfrak{B}}{\partial\tau}+\frac{\partial^2\boldsymbol{\rho}}{\partial t^2}\right)\right]\boldsymbol{r}-r^2\,\frac{\partial}{\partial t}\left(\boldsymbol{\rho}\,\frac{\partial\mathfrak{B}}{\partial\tau}+\frac{\partial^2\boldsymbol{\rho}}{\partial t^2}\right),
\end{aligned}
\right\}
\qquad (1205)
$$

or, using equations (1187) and (1194),

$$
\boldsymbol{r}\times\left[\boldsymbol{r}\times\frac{\partial}{\partial t}\left(\boldsymbol{\rho}\,\frac{\partial\mathfrak{B}}{\partial\tau}+\frac{\partial^2\boldsymbol{\rho}}{\partial t^2}\right)\right]=2\tau\boldsymbol{r}\times\frac{\partial}{\partial t}\left(\boldsymbol{\gamma}\,\frac{\partial\mathfrak{B}}{\partial\tau}+\frac{d^2\boldsymbol{\gamma}}{dt^2}\right). \qquad (1206)
$$

Consequently, equation (1204) can be re-written as

$$\boldsymbol{r} \times \left[\boldsymbol{\delta} \frac{\partial \mathfrak{V}}{\partial \tau} + \frac{d^2 \boldsymbol{\delta}}{dt^2} + 2\tau \frac{\partial}{\partial t} \left(\boldsymbol{\gamma} \frac{\partial \mathfrak{V}}{\partial \tau} + \frac{d^2 \boldsymbol{\gamma}}{dt^2} \right) \right] = 0 . \tag{1207}$$

Again, since $\boldsymbol{\delta}$ and $\boldsymbol{\gamma}$ are two vectors independent of the spatial co-ordinates, equation (1207) implies that

$$\boldsymbol{\delta} \frac{\partial \mathfrak{V}}{\partial \tau} + \frac{d^2 \boldsymbol{\delta}}{dt^2} + 2\tau \frac{\partial}{\partial t} \left(\boldsymbol{\gamma} \frac{\partial \mathfrak{V}}{\partial \tau} + \frac{d^2 \boldsymbol{\gamma}}{dt^2} \right) = 0 . \tag{1208}$$

Remembering that we have explicitly excluded the case of a quasi-elastic field of force, we readily verify that, in order that the two equations (1200) and (1208) be consistent, it is necessary that

$$\boldsymbol{\delta} = 0 . \tag{1209}$$

From equation (1208) it now follows that

$$\frac{\partial}{\partial t} \left(\boldsymbol{\gamma} \frac{\partial \mathfrak{V}}{\partial \tau} + \frac{d^2 \boldsymbol{\gamma}}{dt^2} \right) = 0 . \tag{1210}$$

Equation (1210) can be obtained directly from (1203). For, if $\boldsymbol{\delta} = 0$, this equation reduces to

$$\left. \begin{aligned} \boldsymbol{r} \left(\kappa \frac{\partial^2 \mathfrak{V}}{\partial t \partial \tau} + \tau \frac{d\kappa}{dt} \frac{\partial^2 \mathfrak{V}}{\partial \tau^2} + 2 \frac{d\kappa}{dt} \frac{\partial \mathfrak{V}}{\partial \tau} + \frac{1}{2} \frac{d^3 \kappa}{dt^3} \right) & \\ + \boldsymbol{r} \times \frac{\partial}{\partial t} \left(\boldsymbol{\rho} \frac{\partial \mathfrak{V}}{\partial \tau} + \frac{\partial^2 \boldsymbol{\rho}}{\partial t^2} \right) = 0 . & \end{aligned} \right\} \tag{1211}$$

On multiplying this equation scalarly by r, we obtain

$$\kappa \frac{\partial^2 \mathfrak{V}}{\partial t \partial \tau} + \tau \frac{d\kappa}{dt} \frac{\partial^2 \mathfrak{V}}{\partial \tau^2} + 2 \frac{d\kappa}{dt} \frac{\partial \mathfrak{V}}{\partial \tau} + \frac{1}{2} \frac{d^3 \kappa}{dt^3} = 0 . \tag{1212}$$

Inserting (1212) in equation (1211), we readily recover the result (1210). From equations (1200) and (1210) we now conclude that $\boldsymbol{\gamma}_0$ is a constant vector; in other words,

$$g_0, h_0, f_0, \text{ are all constants} . \tag{1213}$$

Accordingly, we can express the relations (1200″) alternatively in the forms

$$g_{20} = g_0\phi \; ; \qquad h_{20} = h_0\phi \; ; \qquad f_{20} = f_0\phi \; , \tag{1214}$$

where ϕ is an arbitrary function of time; further,

$$D(t) = \frac{q_0}{\phi} \, . \tag{1215}$$

where q_0 is a constant. Equation (1201) now reduces to

$$\frac{\partial \mathfrak{B}}{\partial \tau} = -\frac{\ddot{\phi}}{\phi} + \frac{q_0}{\phi}\frac{1}{\tau^{3/2}} \, . \tag{1216}$$

Again, this restriction on the form of $\mathfrak{B}$ can be avoided only if $\boldsymbol{\gamma} \equiv 0$.

Consider next equation (1212). This equation is identical in form with the equation in a_1 and $\mathfrak{B}$ which we have encountered in Part X (§ 54, Eq. [1095]). Consequently, we can immediately write down the most general solution of this equation. We have (cf. Eqs. [1108] and [1109])

$$\frac{\partial \mathfrak{B}}{\partial \tau} = \left(\frac{q}{\Phi^4} - \frac{\ddot{\Phi}}{\Phi}\right) + \frac{1}{\Phi^4} F_1\left(\frac{\tau}{\Phi^2}\right) , \tag{1217}$$

where q is an arbitrary constant,

$$\kappa = \Phi^2 \, , \tag{1218}$$

and F_1 is an arbitrary function of the argument specified. Equation (1217) can be integrated to give

$$\mathfrak{B} = \left(\frac{q}{\Phi^4} - \frac{\ddot{\Phi}}{\Phi}\right)\tau + \frac{1}{\Phi^2}\mathfrak{B}_1\left(\frac{\tau}{\Phi^2}\right) + \mathfrak{B}_0(t) \, , \tag{1219}$$

where

$$\mathfrak{B}_1(\xi) = \int^{\xi} F_1(\xi)d\xi \tag{1220}$$

and $\mathfrak{B}_0$ a function of time only; further, in (1219) we can regard $\mathfrak{B}_1$ as an arbitrary function of the argument specified.

On the other hand, if $\boldsymbol{\gamma} \not\equiv 0$, $\partial\mathfrak{B}/\partial\tau$ must have the form (1216). This case therefore corresponds to putting

$$q = 0 \; ; \qquad F_1(\xi) = \frac{q_0}{\xi^{3/2}} \tag{1221}$$

in (1217); in (1221), q_0 is a constant. We then have

$$\frac{\partial\mathfrak{B}}{\partial\tau} = -\frac{\dot{\Phi}}{\Phi} + \frac{q_0}{\Phi}\frac{1}{\tau^{3/2}} \,. \tag{1222}$$

To be in agreement with the equations (1215) and (1216) we should have

$$\Phi = \phi \; ; \qquad \kappa = \phi^2 \,. \tag{1223}$$

We can now summarize the results of our present discussion in the following terms:

The most general form of a spherically symmetrical potential function consistent with the compatibility relations (1178) and (1179) is given by

$$\mathfrak{B} = \left(\frac{q}{\Phi^4} - \frac{\ddot{\Phi}}{\Phi}\right)\tau + \frac{1}{\Phi^2}\,\mathfrak{B}_1\left(\frac{\tau}{\Phi^2}\right) + \mathfrak{B}_0(t) \,, \tag{1224}$$

where q is a constant, Φ an arbitrary function of time, and $\mathfrak{B}_1$ and $\mathfrak{B}_0$ arbitrary functions of the arguments specified. For this form of $\mathfrak{B}$ the appropriate solutions for a, b, c, f, g, h, and $\boldsymbol{\Delta}$ are

$$\left.\begin{aligned}
a &= \Phi^2 - h_{40}y^2 - g_{40}z^2 - 2h_{21}yz \,, \\
b &= \Phi^2 - f_{40}z^2 - h_{40}x^2 - 2f_{21}zx \,, \\
c &= \Phi^2 - g_{40}x^2 - f_{40}y^2 - 2g_{21}xy \,, \\
f &= -h_{21}x^2 + f_{21}xy + g_{21}xz + f_{40}yz \,, \\
g &= -f_{21}y^2 + g_{21}yz + h_{21}yx + g_{40}zx \,, \\
h &= -g_{21}z^2 + h_{21}zx + f_{21}zy + h_{40}xy \,,
\end{aligned}\right\} \tag{1225}$$

and

$$\boldsymbol{\Delta} = \Phi\dot{\Phi}\boldsymbol{r} + \boldsymbol{r} \times \boldsymbol{\beta} \,, \tag{1226}$$

where f_{21}, g_{21}, h_{21}, f_{40}, g_{40}, h_{40}, and the vector $\boldsymbol{\beta}$ are all constants.

A special case arises when

$$q = 0 \; ; \qquad \mathfrak{V}_1(\xi) = \frac{q_0}{\xi^{1/2}} \, , \tag{1227}$$

where q_0 is a constant. Under these circumstances the solutions for the coefficients of the velocity ellipsoid are more general and are given by (1183), where

$$g_{20} = g_0 \Phi \; ; \qquad h_{20} = h_0 \Phi \; ; \qquad f_{20} = f_0 \Phi \, , \tag{1228}$$

g_0, h_0, and f_0 being constants; further, the solution for $\boldsymbol{\Delta}$ appropriate to this special case is

$$\boldsymbol{\Delta} = \Phi(\boldsymbol{r} \times \boldsymbol{\rho}_0) + \Phi\dot{\Phi}\boldsymbol{r} + \boldsymbol{r} \times \boldsymbol{\beta} \, , \tag{1229}$$

where

$$\boldsymbol{\rho}_0 = \boldsymbol{\gamma}_0 \times \boldsymbol{r} \; ; \qquad \boldsymbol{\gamma}_0 = (g_0, h_0, f_0) \, . \tag{1230}$$

We shall complete our discussion of the spherically symmetrical case by deriving explicitly the solutions for χ:

(i) *The general case.*—The most general form of the spherically symmetrical potential function consistent with the integrability conditions is given in equation (1224). For this form of $\mathfrak{V}$ we have

$$\frac{\partial \mathfrak{V}}{\partial \tau} = \left(\frac{q}{\Phi^4} - \frac{\ddot{\Phi}}{\Phi}\right) + \frac{1}{\Phi^4} \mathfrak{V}_1' \, , \tag{1231}$$

where a prime denotes differentiation with respect to the argument τ/Φ^2 of $\mathfrak{V}_1$. Further, for this case we have (cf. Eqs. [1189] and [1226])

$$\boldsymbol{\Delta} = \Phi\dot{\Phi}\boldsymbol{r} + \boldsymbol{r} \times \boldsymbol{\beta} \tag{1232}$$

and

$$A\boldsymbol{r} = \Phi^2 \boldsymbol{r} \; ; \qquad \boldsymbol{\Delta} \cdot \boldsymbol{r} = 2\Phi\dot{\Phi}\tau \, . \tag{1233}$$

Substituting from equations (1231)–(1233) in the compatibility relations (1178) and (1179), we find, after some minor reductions, that

$$\left(\frac{q}{\Phi^2} + \dot{\Phi}^2 + \frac{1}{\Phi^2} \mathfrak{V}_1'\right) \boldsymbol{r} = -\tfrac{1}{2} \operatorname{grad} \chi \tag{1234}$$

and

$$2\left(\frac{q}{\Phi^3}-\ddot{\Phi}\right)\dot{\Phi}\tau+2\,\frac{\dot{\Phi}\tau}{\Phi^3}\,\mathfrak{B}_1'=\frac{1}{2}\,\frac{\partial\chi}{\partial t}\,. \tag{1235}$$

According to equation (1234),

$$\boldsymbol{r}\times\operatorname{grad}\chi=0\;; \tag{1236}$$

in other words, χ is a spherically symmetrical function. Hence,

$$\operatorname{grad}\chi=\boldsymbol{r}\,\frac{\partial\chi}{\partial\tau}\,. \tag{1237}$$

Equation (1234) can now be re-written as

$$-\frac{1}{2}\,\frac{\partial\chi}{\partial\tau}=\frac{q}{\Phi^2}+\dot{\Phi}^2+\frac{\partial\mathfrak{B}_1}{\partial\tau}\,. \tag{1238}$$

Also, since

$$\frac{\partial}{\partial t}\,\mathfrak{B}_1\left(\frac{\tau}{\Phi^2}\right)=-\,2\,\frac{\dot{\Phi}\tau}{\Phi^3}\,\mathfrak{B}_1'\,, \tag{1239}$$

equation (1235) can be expressed alternatively as

$$-\frac{1}{2}\,\frac{\partial\chi}{\partial t}=2\left(-\frac{q}{\Phi^3}\,\dot{\Phi}+\dot{\Phi}\ddot{\Phi}\right)\tau+\frac{\partial\mathfrak{B}_1}{\partial t}\,. \tag{1240}$$

From equations (1238) and (1240) we conclude that

$$-\tfrac{1}{2}\chi=\left(\frac{q}{\Phi^2}+\dot{\Phi}^2\right)\tau+\mathfrak{B}_1\left(\frac{\tau}{\Phi^2}\right)+\text{constant}\;. \tag{1241}$$

(ii) *The special case.*—We now have (cf. Eq. [1222])

$$\frac{\partial\mathfrak{B}}{\partial\tau}=-\frac{\ddot{\Phi}}{\Phi}+\frac{q_0}{\Phi}\,\frac{1}{\tau^{3/2}}\,. \tag{1242}$$

Further, according to equations (1187), (1189), (1229), and (1230), we have

$$\left.\begin{aligned}\boldsymbol{A}\boldsymbol{r}&=\Phi^2\boldsymbol{r}+(\boldsymbol{r}\times\boldsymbol{\rho}_0)\Phi\,,\\ \boldsymbol{\Delta}\cdot\boldsymbol{r}&=2\tau\Phi\dot{\Phi}\,.\end{aligned}\right\} \tag{1243}$$

Substituting from (1243) in equations (1178) and (1179), we find (after some rearranging of the terms) that

$$\Phi^2 \boldsymbol{r} + \frac{q_0 \Phi}{\tau^{3/2}} \boldsymbol{r} + \frac{q_0}{\tau^{3/2}} (\boldsymbol{r} \times \boldsymbol{\rho}_0) = -\tfrac{1}{2} \operatorname{grad} \chi \tag{1244}$$

and

$$-2\Phi\ddot{\Phi}\tau + 2q_0\ddot{\Phi} \frac{1}{\tau^{1/2}} = \frac{1}{2} \frac{\partial \chi}{\partial t} . \tag{1245}$$

From equation (1245) it readily follows that

$$\tfrac{1}{2}\chi = -\dot{\Phi}^2\tau + 2q_0\dot{\Phi} \frac{1}{\tau^{1/2}} + \tfrac{1}{2}\chi_1(x, y, z) , \tag{1246}$$

where χ_1 is a function of the spatial co-ordinates only. According to equation (1246),

$$\tfrac{1}{2} \operatorname{grad} \chi = -\dot{\Phi}^2 \boldsymbol{r} - \frac{q_0}{\tau^{3/2}} \dot{\Phi} \boldsymbol{r} + \tfrac{1}{2} \operatorname{grad} \chi_1 . \tag{1247}$$

Substituting from (1247) in equation (1244), we find

$$-\tfrac{1}{2} \operatorname{grad} \chi_1 = \frac{q_0}{\tau^{3/2}} (\boldsymbol{r} \times \boldsymbol{\rho}_0) . \tag{1248}$$

Now, according to equation (1230),

$$\boldsymbol{r} \times \boldsymbol{\rho}_0 = -\boldsymbol{r} \times (\boldsymbol{r} \times \boldsymbol{\gamma}_0) ; \tag{1249}$$

or, using the formula for the vector triple product, we have

$$\boldsymbol{r} \times \boldsymbol{\rho}_0 = -(\boldsymbol{r} \cdot \boldsymbol{\gamma}_0)\boldsymbol{r} + 2\tau\boldsymbol{\gamma}_0 . \tag{1250}$$

Hence, equation (1248) becomes

$$-\tfrac{1}{2} \operatorname{grad} \chi_1 = q_0 \left[\frac{2}{\tau^{1/2}} \boldsymbol{\gamma}_0 - \frac{1}{\tau^{3/2}} (\boldsymbol{r} \cdot \boldsymbol{\gamma}_0)\boldsymbol{r} \right] . \tag{1251}$$

From equation (1251) it readily follows that

$$-\tfrac{1}{2}\chi_1 = \frac{2q_0}{\tau^{1/2}} (\boldsymbol{\gamma}_0 \cdot \boldsymbol{r}) + \text{constant} . \tag{1252}$$

Finally, combining equations (1246) and (1252), we have

$$-\tfrac{1}{2}\chi = \Phi^2\tau + \frac{2q_0}{\tau^{1/2}}[(\boldsymbol{\gamma}_0 \cdot \boldsymbol{r}) - \Phi] + \text{constant} . \qquad (1253)$$

In concluding this discussion we may note that, just as in the two-dimensional problem, the case in which $\mathfrak{V}$ arises from the superposition of *constant* quasi-elastic and inverse cube fields of forces has some interesting features. For this case there exist solutions for a, b, and c which depend explicitly on the time. Since the discussion of this case can be carried out exactly on the lines of § 55, we shall not go into the details here; we may, however, note that, as before, the solution for Φ^2 is found to take the form (cf. Eq. [1135])

$$\Phi^2 = a_0 + a_{01} \sin 2q_1(t - t_0) . \qquad (1254)$$

This completes our discussion of the spherically symmetrical case.

59. *Examples of stellar systems having axial symmetries and in non-steady states.*—We have already seen, in Part VII, § 33, that under *steady-state circumstances* the integrability conditions (780)–(782) are "trivially" satisfied in the following two cases:

(i) With no restriction on $\mathfrak{V}(\bar{\omega}, z)$, equations (780)–(782) are satisfied if

$$f = g = h = 0 ; \qquad a = c . \qquad (1255)$$

Under these circumstances we can write (cf. Eq. [784])

$$a = \kappa_1 ; \qquad b = \kappa_1 + \kappa_2\bar{\omega}^2 ; \qquad c = \kappa_1 , \qquad (1256)$$

where κ_1 and κ_2 are constants. Also, the solutions appropriate for the Δ's are (cf. Eq. [777])

$$\Delta_1 = 0 ; \qquad \Delta_2 = p\bar{\omega} ; \qquad \Delta_3 = 0 , \qquad (1257)$$

where p is a constant. The motions of the local centroids which correspond to (1257) are

$$\Pi_0 = 0 ; \qquad \Theta_0 = \frac{p\bar{\omega}}{\kappa_1 + \kappa_2\bar{\omega}^2} ; \qquad Z_0 = 0 . \qquad (1258)$$

(ii) If we assume that

$$\frac{\partial^2 \mathfrak{B}}{\partial \bar{\omega} \partial z} = 0 , \tag{1259}$$

then we have the more general solution

$$a = \kappa_1 ; \qquad b = \kappa_1 + \kappa_2 \bar{\omega}^2 ; \qquad c = \kappa_3 , \tag{1260}$$

where κ_1, κ_2, and κ_3 are constants. The solutions for the Δ's are the same as in case (i) above.

The two cases enumerated above, as already stated, are valid under steady-state conditions. We shall now consider the analogues of these two cases under nonsteady-state conditions.

(A) Corresponding to case (i) above, we have (in a Cartesian system of co-ordinates[49])

$$\left.\begin{aligned} a &= \kappa_1 + \kappa_2 y^2 ; & b &= \kappa_1 + \kappa_2 x^2 , \\ c &= \kappa_1 ; & h &= -\kappa_2 xy , \end{aligned}\right\} \tag{1261}$$

where κ_1 is an arbitrary function of the time and κ_2 is a constant. In writing down the solution appropriate for the Δ's, we shall be guided by the results we have obtained for the corresponding two-dimensional problem: In Part X we have shown that for the most general potential function $\mathfrak{B}(\bar{\omega}, t)$ (in the plane) which is consistent with the integrability conditions the motions of the local centroids correspond to the superposition of a K-term proportional to the distance and a differential rotation term according to (1258) (cf. § 54, Eq. [1116]). We shall therefore assume that

$$\left.\begin{aligned} \Delta_1 &= \frac{1}{2} \frac{d\kappa_1}{dt} x + \beta y , \\ \Delta_2 &= \frac{1}{2} \frac{d\kappa_1}{dt} y - \beta x , \\ \Delta_3 &= \frac{1}{2} \frac{d\kappa_1}{dt} z , \end{aligned}\right\} \tag{1262}$$

where β is a constant.[50]

[49] The solution for the cases (i) and (ii) have been given in a cylindrical system of co-ordinates.

[50] The constancy of β, here assumed, is equivalent to the constancy of p, which we proved in Part X (cf. Eq. [995]) when considering the corresponding two-dimensional problem.

(B) If, as in case (ii),

$$\frac{\partial^2 \mathfrak{V}}{\partial \bar{\omega} \partial z} = 0 , \tag{1263}$$

then we shall assume that

$$\left.\begin{aligned} a &= \kappa_1 + \kappa_2 y^2 ; \qquad & b &= \kappa_1 + \kappa_2 x^2 , \\ c &= \kappa_3 ; & h &= -\kappa_2 xy , \end{aligned}\right\} \tag{1264}$$

where κ_1 and κ_3 are two arbitrary functions of time and κ_2 is a constant. Further,

$$\left.\begin{aligned} \Delta_1 &= \frac{1}{2}\frac{d\kappa_1}{dt} x + \beta y , \\ \Delta_2 &= \frac{1}{2}\frac{d\kappa_1}{dt} y - \beta x , \\ \Delta_3 &= \frac{1}{2}\frac{d\kappa_3}{dt} z , \end{aligned}\right\} \tag{1265}$$

We shall now proceed to the consideration of the compatibility relations for the two cases enumerated above:

Case (A).—The equations we have to consider are

$$\left.\begin{aligned} (\kappa_1 + \kappa_2 y^2)\frac{\partial \mathfrak{V}}{\partial x} - \kappa_2 xy \frac{\partial \mathfrak{V}}{\partial y} + \tfrac{1}{2} x \frac{d^2\kappa_1}{dt^2} &= -\frac{1}{2}\frac{\partial \chi}{\partial x} , \\ (\kappa_1 + \kappa_2 x^2)\frac{\partial \mathfrak{V}}{\partial y} - \kappa_2 xy \frac{\partial \mathfrak{V}}{\partial x} + \tfrac{1}{2} y \frac{d^2\kappa_1}{dt^2} &= -\frac{1}{2}\frac{\partial \chi}{\partial y} , \\ \kappa_1 \frac{\partial \mathfrak{V}}{\partial z} + \tfrac{1}{2} z \frac{d^2\kappa_1}{dt^2} &= -\frac{1}{2}\frac{\partial \chi}{\partial z} , \\ \left(\frac{1}{2}\frac{d\kappa_1}{dt} x + \beta y\right)\frac{\partial \mathfrak{V}}{\partial x} + \left(\frac{1}{2}\frac{d\kappa_1}{dt} y - \beta x\right)\frac{\partial \mathfrak{V}}{\partial y} + \frac{1}{2}\frac{d\kappa_1}{dt} z \frac{\partial \mathfrak{V}}{\partial z} &= \frac{1}{2}\frac{\partial \chi}{\partial t} , \end{aligned}\right\} \tag{1266}$$

where, according to our assumption,

$$\mathfrak{V} \equiv \mathfrak{V}(\bar{\omega}, z, t) . \tag{1267}$$

Let

$$\tau = \tfrac{1}{2}(x^2 + y^2) ; \qquad \zeta = \tfrac{1}{2} z^2 . \tag{1268}$$

In terms of these variables equations (1266) take the simpler forms

$$\kappa_1 x \frac{\partial \mathfrak{V}}{\partial \tau} + \tfrac{1}{2} x \frac{d^2\kappa_1}{dt^2} = -\frac{1}{2}\frac{\partial \chi}{\partial x}, \tag{1269}$$

$$\kappa_1 y \frac{\partial \mathfrak{V}}{\partial \tau} + \tfrac{1}{2} y \frac{d^2\kappa_1}{dt^2} = -\frac{1}{2}\frac{\partial \chi}{\partial y}, \tag{1270}$$

$$\kappa_1 \frac{\partial \mathfrak{V}}{\partial \zeta} + \frac{1}{2}\frac{d^2\kappa_1}{dt^2} = -\frac{1}{2}\frac{\partial \chi}{\partial \zeta}, \tag{1271}$$

and

$$\frac{d\kappa_1}{dt}\left(\tau \frac{\partial \mathfrak{V}}{\partial \tau} + \zeta \frac{\partial \mathfrak{V}}{\partial \zeta}\right) = \frac{1}{2}\frac{\partial \chi}{\partial t}. \tag{1272}$$

From equations (1269) and (1270) we infer that

$$y\frac{\partial \chi}{\partial x} = x\frac{\partial \chi}{\partial y}. \tag{1273}$$

Hence,

$$\chi \equiv \chi(\tau, \zeta, t). \tag{1274}$$

Thus, the two equations (1269) and (1270) are equivalent to the single equation

$$\kappa_1 \frac{\partial \mathfrak{V}}{\partial \tau} + \frac{1}{2}\frac{d^2\kappa_1}{dt^2} = -\frac{1}{2}\frac{\partial \chi}{\partial \tau}. \tag{1275}$$

The integrability condition resulting from equations (1271) and (1275) is clearly satisfied. On the other hand, from equations (1272) and (1275) we have

$$\frac{\partial}{\partial t}\left(\kappa_1 \frac{\partial \mathfrak{V}}{\partial \tau} + \frac{1}{2}\frac{d^2\kappa_1}{dt^2}\right) + \frac{d\kappa_1}{dt}\frac{\partial}{\partial \tau}\left(\tau \frac{\partial \mathfrak{V}}{\partial \tau} + \zeta \frac{\partial \mathfrak{V}}{\partial \zeta}\right) = 0; \tag{1276}$$

or, after performing the differentiations, we have

$$\kappa_1 \frac{\partial^2 \mathfrak{V}}{\partial t \partial \tau} + \tau \frac{d\kappa_1}{dt}\frac{\partial^2 \mathfrak{V}}{\partial \tau^2} + \zeta \frac{d\kappa_1}{dt}\frac{\partial^2 \mathfrak{V}}{\partial \tau \partial \zeta} + 2\frac{d\kappa_1}{dt}\frac{\partial \mathfrak{V}}{\partial \tau} + \frac{1}{2}\frac{d^3\kappa_1}{dt^3} = 0. \tag{1277}$$

Similarly, from equations (1271) and (1272) we obtain

$$\kappa_1 \frac{\partial^2 \mathfrak{V}}{\partial t \partial \zeta} + \zeta \frac{d\kappa_1}{dt} \frac{\partial^2 \mathfrak{V}}{\partial \zeta^2} + \tau \frac{d\kappa_1}{dt} \frac{\partial^2 \mathfrak{V}}{\partial \tau \partial \zeta} + 2 \frac{d\kappa_1}{dt} \frac{\partial \mathfrak{V}}{\partial \zeta} + \frac{1}{2} \frac{d^3\kappa_1}{dt^3} = 0 . \tag{1278}$$

Equations (1277) and (1278) can be written alternatively in the forms

$$\left.\begin{aligned} \frac{\partial}{\partial \tau} \left(\frac{d\kappa_1}{dt} \mathfrak{V} + \kappa_1 \frac{\partial \mathfrak{V}}{\partial t} + \tau \frac{d\kappa_1}{dt} \frac{\partial \mathfrak{V}}{\partial \tau} + \zeta \frac{d\kappa_1}{dt} \frac{\partial \mathfrak{V}}{\partial \zeta} + \frac{1}{2} \frac{d^3\kappa_1}{dt^3} \tau \right) &= 0 , \\ \frac{\partial}{\partial \zeta} \left(\frac{d\kappa_1}{dt} \mathfrak{V} + \kappa_1 \frac{\partial \mathfrak{V}}{\partial t} + \tau \frac{d\kappa_1}{dt} \frac{\partial \mathfrak{V}}{\partial \tau} + \zeta \frac{d\kappa_1}{dt} \frac{\partial \mathfrak{V}}{\partial \zeta} + \frac{1}{2} \frac{d^3\kappa_1}{dt^3} \zeta \right) &= 0 . \end{aligned}\right\} \tag{1279}$$

From the foregoing equations it readily follows that

$$\kappa_1 \frac{\partial \mathfrak{V}}{\partial t} + \frac{d\kappa_1}{dt} \left(\tau \frac{\partial \mathfrak{V}}{\partial \tau} + \zeta \frac{\partial \mathfrak{V}}{\partial \zeta} + \mathfrak{V} \right) = -\frac{1}{2} \frac{d^3\kappa_1}{dt^3} (\tau + \zeta) + F^*(t) , \tag{1280}$$

where $F^*(t)$ is a function of t only. Equation (1280) is readily seen to be a *nonhomogeneous* linear partial differential equation for $\kappa_1\mathfrak{V}$, for we can re-write this equation as

$$\left.\begin{aligned} \frac{\partial}{\partial t} (\kappa_1 \mathfrak{V}) + \frac{1}{\kappa_1} \frac{d\kappa_1}{dt} \left\{ \tau \frac{\partial}{\partial \tau} (\kappa_1 \mathfrak{V}) + \zeta \frac{\partial}{\partial \zeta} (\kappa_1 \mathfrak{V}) \right\} & \\ = -\tfrac{1}{2}(\tau + \zeta) \frac{d^3\kappa_1}{dt^3} + F^*(t) &. \end{aligned}\right\} \tag{1281}$$

To solve this equation, differentiate it partially with respect to τ and ζ. We obtain

$$\left.\begin{aligned} \frac{\partial^3}{\partial t \partial \tau \partial \zeta} (\kappa_1 \mathfrak{V}) + \frac{1}{\kappa_1} \frac{d\kappa_1}{dt} \left\{ \tau \frac{\partial^3}{\partial \tau^2 \partial \zeta} (\kappa_1 \mathfrak{V}) + \zeta \frac{\partial^3}{\partial \tau \partial \zeta^2} (\kappa_1 \mathfrak{V}) \right. & \\ \left. + 2 \frac{\partial^2}{\partial \tau \partial \zeta} (\kappa_1 \mathfrak{V}) \right\} = 0 &. \end{aligned}\right\} \tag{1282}$$

Let

$$\psi = \frac{\partial^2}{\partial \tau \partial \zeta} (\kappa_1 \mathfrak{V}) = \kappa_1 \frac{\partial^2 \mathfrak{V}}{\partial \tau \partial \zeta} . \tag{1283}$$

In terms of ψ equation (1282) can be re-written as

$$\frac{\partial \psi}{\partial t} + \frac{1}{\kappa_1} \frac{d\kappa_1}{dt} \left(\tau \frac{\partial \psi}{\partial \tau} + \zeta \frac{\partial \psi}{\partial \zeta} + 2\psi \right) = 0 , \tag{1284}$$

or alternatively as

$$\frac{\partial \psi}{\partial t} + \frac{1}{\kappa_1} \frac{d\kappa_1}{dt} \left\{ \frac{\partial}{\partial \tau} (\psi \tau) + \frac{\partial}{\partial \zeta} (\psi \zeta) \right\} = 0 . \tag{1285}$$

Multiplying the foregoing equation by $\tau\zeta$, we have

$$\frac{\partial}{\partial t} (\psi \tau \zeta) + \frac{1}{\kappa_1} \frac{d\kappa_1}{dt} \left\{ \tau \frac{\partial}{\partial \tau} (\psi \tau \zeta) + \zeta \frac{\partial}{\partial \zeta} (\psi \tau \zeta) \right\} = 0 , \tag{1286}$$

which is a *homogeneous* partial differential equation of the Lagrangian type for $\psi\tau\zeta$. The appropriate subsidiary equations are

$$dt = \frac{d\tau}{\dfrac{\tau}{\kappa_1} \dfrac{d\kappa_1}{dt}} = \frac{d\zeta}{\dfrac{\zeta}{\kappa_1} \dfrac{d\kappa_1}{dt}} . \tag{1287}$$

The foregoing equations are equivalent to

$$\frac{d\kappa_1}{\kappa_1} = \frac{d\tau}{\tau} = \frac{d\zeta}{\zeta} . \tag{1288}$$

Two independent integrals of (1288) are

$$\frac{\tau}{\kappa_1} = \text{constant} ; \qquad \frac{\zeta}{\kappa_1} = \text{constant} . \tag{1289}$$

Consequently,

$$\kappa_1 \tau \zeta \frac{\partial^2 \mathfrak{P}}{\partial \tau \partial \zeta} = \text{Function} \left(\frac{\tau}{\kappa_1} , \frac{\zeta}{\kappa_1} \right) , \tag{1290}$$

or, alternatively,

$$\frac{\partial^2 \mathfrak{P}}{\partial \tau \partial \zeta} = \frac{1}{\kappa_1^3} F \left(\frac{\tau}{\kappa_1} , \frac{\zeta}{\kappa_1} \right) , \tag{1291}$$

where F is an arbitrary function of the arguments specified. Integrating equation (1291) with respect to ζ and τ, we obtain, respectively,

$$\left.\begin{aligned}\frac{\partial\mathfrak{V}}{\partial\tau}&=\frac{\partial}{\partial\tau}\,\mathfrak{V}_1(\tau,t)+\frac{1}{\kappa_1^3}\int^{\zeta}F\left(\frac{\tau}{\kappa_1},\frac{\zeta}{\kappa_1}\right)d\zeta\,,\\ \frac{\partial\mathfrak{V}}{\partial\zeta}&=\frac{\partial}{\partial\tau}\,\mathfrak{V}_2(\zeta,t)+\frac{1}{\kappa_1^3}\int^{\tau}F\left(\frac{\tau}{\kappa_1},\frac{\zeta}{\kappa_1}\right)d\tau\,,\end{aligned}\right\}\qquad(1292)$$

where $\mathfrak{V}_1$ and $\mathfrak{V}_2$ are (as indicated) independent of ζ and τ, respectively. Substituting for $\partial\mathfrak{V}/\partial\tau$ and $\partial\mathfrak{V}/\partial\zeta$ according to (1292) in the equations (1277) and (1278), we obtain, respectively,

$$\kappa_1\frac{\partial^2\mathfrak{V}_1}{\partial t\partial\tau}+\tau\frac{d\kappa_1}{dt}\frac{\partial^2\mathfrak{V}_1}{\partial\tau^2}+2\frac{d\kappa_1}{dt}\frac{\partial\mathfrak{V}_1}{\partial\tau}+\frac{1}{2}\frac{d^3\kappa_1}{dt^3}=0\qquad(1293)$$

and

$$\kappa_1\frac{\partial^2\mathfrak{V}_2}{\partial t\partial\zeta}+\zeta\frac{d\kappa_1}{dt}\frac{\partial^2\mathfrak{V}_2}{\partial\zeta^2}+2\frac{d\kappa_1}{dt}\frac{\partial\mathfrak{V}_2}{\partial\zeta}+\frac{1}{2}\frac{d^3\kappa_1}{dt^3}=0\,.\qquad(1294)$$

The foregoing equations are identical in forms with the equation (1095) we have already encountered in Part X, § 54. We can therefore write (cf. Eqs. [1108] and [1109])

$$\left.\begin{aligned}\frac{\partial\mathfrak{V}_1}{\partial\tau}&=\left(\frac{q_1}{\phi^4}-\frac{\ddot\phi}{\phi}\right)+\frac{1}{\phi^4}F_1\left(\frac{\tau}{\phi^2}\right),\\ \frac{\partial\mathfrak{V}_2}{\partial\zeta}&=\left(\frac{q_2}{\phi^4}-\frac{\ddot\phi}{\phi}\right)+\frac{1}{\phi^4}F_2\left(\frac{\zeta}{\phi^2}\right),\end{aligned}\right\}\qquad(1295)$$

where

$$\kappa_1=\phi^2\qquad(1296)$$

and where F_1 and F_2 are arbitrary functions of the arguments specified and q_1 and q_2 are constants. Combining the equations (1292) and (1295), we obtain

$$\left.\begin{aligned}\frac{\partial\mathfrak{V}}{\partial\tau}&=\left(\frac{q_1}{\phi^4}-\frac{\ddot\phi}{\phi}\right)+\frac{1}{\phi^4}F_1\left(\frac{\tau}{\phi^2}\right)+\frac{1}{\phi^4}\int^{\zeta/\phi^2}F\left(\frac{\tau}{\phi^2},\frac{\zeta}{\phi^2}\right)\frac{d\zeta}{\phi^2}\,,\\ \frac{\partial\mathfrak{V}}{\partial\zeta}&=\left(\frac{q_2}{\phi^4}-\frac{\ddot\phi}{\phi}\right)+\frac{1}{\phi^4}F_2\left(\frac{\zeta}{\phi^2}\right)+\frac{1}{\phi^4}\int^{\tau/\phi^2}F\left(\frac{\tau}{\phi^2},\frac{\zeta}{\phi^2}\right)\frac{d\tau}{\phi^2}\,.\end{aligned}\right\}\qquad(1297)$$

From (1297) we readily have

$$\left.\begin{aligned}\mathfrak{V} = \left(\frac{q_1}{\phi^4} - \frac{\ddot{\phi}}{\phi}\right)\tau + \left(\frac{q_2}{\phi^4} - \frac{\ddot{\phi}}{\phi}\right)\zeta + \frac{1}{\phi^2}\int^{\tau/\phi^2} F_1\left(\frac{\tau}{\phi^2}\right)\frac{d\tau}{\phi^2} \\ + \frac{1}{\phi^2}\int^{\zeta/\phi^2} F_2\left(\frac{\zeta}{\phi^2}\right)\frac{d\zeta}{\phi^2} + \frac{1}{\phi^2}\int^{\tau/\phi^2}\int^{\zeta/\phi^2} F\left(\frac{\tau}{\phi^2}, \frac{\zeta}{\phi^2}\right)\frac{d\tau}{\phi^2}\frac{d\zeta}{\phi^2}.\end{aligned}\right\} \quad (1298)$$

Since F, F_1, and F_2 are all arbitrary functions of the arguments indicated, equation (1298) is clearly equivalent to

$$\mathfrak{V} = \left(\frac{q_1}{\phi^4} - \frac{\ddot{\phi}}{\phi}\right)\tau + \left(\frac{q_2}{\phi^4} - \frac{\ddot{\phi}}{\phi}\right)\zeta + \frac{1}{\phi^2}\mathfrak{V}^*\left(\frac{\tau}{\phi^2}, \frac{\zeta}{\phi^2}\right) + \mathfrak{V}_0(t), \quad (1299)$$

where $\mathfrak{V}^*$ and $\mathfrak{V}_0$ are arbitrary functions of the arguments specified.[51]

Thus, the necessary and sufficient condition that the equations (1271), (1272), and (1275) be compatible is that $\mathfrak{V}$ be of the form given by (1299). Substituting now this expression for $\mathfrak{V}$ in the equations (1271), (1272), and (1275) and remembering that $\kappa_1 = \phi^2$, we obtain

$$\left.\begin{aligned}2\tau\phi\dot{\phi}\left(\frac{q_1}{\phi^4} - \frac{\ddot{\phi}}{\phi} + \frac{1}{\phi^4}\frac{\partial\mathfrak{V}^*}{\partial(\tau/\phi^2)}\right) \qquad\qquad \\ + 2\zeta\phi\dot{\phi}\left(\frac{q_2}{\phi^4} - \frac{\ddot{\phi}}{\phi} + \frac{1}{\phi^4}\frac{\partial\mathfrak{V}^*}{\partial(\zeta/\phi^2)}\right) = \frac{1}{2}\frac{\partial\chi}{\partial t}, \\ \phi^2\left(\frac{q_1}{\phi^4} - \frac{\ddot{\phi}}{\phi} + \frac{1}{\phi^4}\frac{\partial\mathfrak{V}^*}{\partial(\tau/\phi^2)}\right) + \dot{\phi}^2 + \phi\ddot{\phi} = -\frac{1}{2}\frac{\partial\chi}{\partial\tau}. \\ \phi^2\left(\frac{q_2}{\phi^4} - \frac{\ddot{\phi}}{\phi} + \frac{1}{\phi^4}\frac{\partial\mathfrak{V}^*}{\partial(\zeta/\phi^2)}\right) + \dot{\phi}^2 + \phi\ddot{\phi} = -\frac{1}{2}\frac{\partial\chi}{\partial\zeta}.\end{aligned}\right\} \quad (1300)$$

On the other hand,

$$\left.\begin{aligned}\frac{\partial\mathfrak{V}^*}{\partial t} = -2\frac{\dot{\phi}}{\phi^3}\left(\tau\frac{\partial\mathfrak{V}^*}{\partial(\tau/\phi^2)} + \zeta\frac{\partial\mathfrak{V}^*}{\partial(\zeta/\phi^2)}\right), \\ \frac{\partial\mathfrak{V}^*}{\partial\tau} = \frac{1}{\phi^2}\frac{\partial\mathfrak{V}^*}{\partial(\tau/\phi^2)}; \qquad \frac{\partial\mathfrak{V}^*}{\partial\zeta} = \frac{1}{\phi^2}\frac{\partial\mathfrak{V}^*}{\partial(\zeta/\phi^2)}.\end{aligned}\right\} \quad (1301)$$

[51] Eq. (1299) therefore represents the most general solution of the nonhomogeneous Eq. (1281). We shall consider later (§ 60) the mathematical aspects of this situation.

Using (1301), equations (1300) can be reduced to

$$\left.\begin{aligned}
-\frac{1}{2}\frac{\partial\chi}{\partial t} &= 2\tau\left(-\frac{q_1}{\phi^3}\dot{\phi}+\dot{\phi}\ddot{\phi}\right)+2\zeta\left(-\frac{q_2}{\phi^3}\dot{\phi}+\dot{\phi}\ddot{\phi}\right)+\frac{\partial\mathfrak{B}^*}{\partial t}\,,\\
-\frac{1}{2}\frac{\partial\chi}{\partial\tau} &= \frac{q_1}{\phi^2}+\dot{\phi}^2+\frac{\partial\mathfrak{B}^*}{\partial\tau}\,,\\
-\frac{1}{2}\frac{\partial\chi}{\partial\zeta} &= \frac{q_2}{\phi^2}+\dot{\phi}^2+\frac{\partial\mathfrak{B}^*}{\partial\zeta}\,.
\end{aligned}\right\}\qquad(1302)$$

From the foregoing equations it readily follows that

$$\left.\begin{aligned}
-\tfrac{1}{2}\chi = \left(\frac{q_1}{\phi^2}+\dot{\phi}^2\right)\tau+\left(\frac{q_2}{\phi^2}+\dot{\phi}^2\right)\zeta+\mathfrak{B}^*\left(\frac{\tau}{\phi^2},\frac{\zeta}{\phi^2}\right)&\\
+\text{ constant}\,.&
\end{aligned}\right\}\qquad(1303)$$

This completes our discussion of the equations (1271), (1272), and (1275).

We shall now examine the physical characteristics of the stellar system under consideration somewhat more closely. The solutions for the coefficients of the velocity ellipsoid and the Δ's are

$$\left.\begin{aligned}
a = \phi^2+\kappa_2 y^2\,;\qquad b = \phi^2+\kappa_2 x^2\,;\qquad c = \phi^2\,,\\
f = g = 0\,;\qquad h = -\kappa_2 xy\,,
\end{aligned}\right\}\qquad(1304)$$

and

$$\left.\begin{aligned}
\Delta_1 &= aU_0+hV_0 = \phi\dot{\phi}x+\beta y\,,\\
\Delta_2 &= hU_0+bV_0 = \phi\dot{\phi}y-\beta x\,,\\
\Delta_3 &= cW_0 = \phi\dot{\phi}z\,.
\end{aligned}\right\}\qquad(1305)$$

From the equations (1304) and (1305) we readily obtain

$$\left.\begin{aligned}
U_0 &= \frac{\dot{\phi}}{\phi}x+\frac{\beta y}{\phi^2+\kappa_2(x^2+y^2)}\,,\\
V_0 &= \frac{\dot{\phi}}{\phi}y-\frac{\beta x}{\phi^2+\kappa_2(x^2+y^2)}\,,\\
W_0 &= \frac{\dot{\phi}}{\phi}z\,.
\end{aligned}\right\}\qquad(1306)$$

Let Π_0, Θ_0, and Z_0 denote the components of motion of the local centroid along the radial, transverse, and z-directions, respectively. Then

$$\left.\begin{aligned} \Pi_0 &= \frac{1}{\varpi}(yV_0 + xU_0) = \frac{\dot\phi}{\phi}\varpi\,, \\ \Theta_0 &= \frac{1}{\varpi}(xV_0 - yU_0) = -\frac{\beta\varpi}{\phi^2 + \kappa_2\varpi^2}\,, \\ Z_0 &= W_0 = \frac{\dot\phi}{\phi}z\,. \end{aligned}\right\} \tag{1307}$$

Thus, the field of differential motions consists of a uniform expansion (or contraction depending on the sign of $\dot\phi$) and a simple rotation about the z-axis. Further, it is seen that the solution (1304) for the coefficients of the velocity ellipsoid predicts no deviation of the vertex (cf. § 9).

Finally, let us consider the solution for the density function σ. By definition (Eq. [935]) we have

$$-\chi = Q_0 + \sigma\,, \tag{1308}$$

where

$$Q_0 = \Delta_1 U_0 + \Delta_2 V_0 + \Delta_3 W_0\,. \tag{1309}$$

According to equations (1305) and (1306), we have for the case under consideration

$$Q_0 = \left(\dot\phi^2 + \frac{\beta^2}{\phi^2 + \kappa_2\varpi^2}\right)\varpi^2 + \dot\phi^2 z^2\,. \tag{1310}$$

Combining equations (1303), (1308), and (1310), we find that

$$\left.\begin{aligned} \sigma + \dot\phi^2(\varpi^2 + z^2) + \frac{\beta^2\varpi^2}{\phi^2 + \kappa_2\varpi^2} \\ = \frac{q_1}{\phi^2}\varpi^2 + \frac{q_2}{\phi^2}z^2 + \dot\phi^2(\varpi^2 + z^2) + 2\mathfrak{V}^* + \text{constant}\,. \end{aligned}\right\} \tag{1311}$$

Hence,

$$\left.\begin{aligned}\sigma = \frac{1}{\phi^2}\left\{\left(q_1 - \frac{\beta^2\phi^2}{\phi^2 + \kappa_2\bar{\omega}^2}\right)\bar{\omega}^2 + q_2 z^2\right\} \\ + 2\mathfrak{V}^*\left(\frac{\tau}{\phi^2}, \frac{\zeta}{\phi^2}\right) + \text{constant}\,.\end{aligned}\right\} \tag{1312}$$

If we now assume that

$$\Psi(Q + \sigma) = e^{-(Q+\sigma)}\,, \tag{1313}$$

then the number $\mathfrak{N}$ of stars per unit volume is given by

$$\mathfrak{N} = \frac{\pi^{3/2} e^{-\sigma}}{\sqrt{D}}\,, \tag{1314}$$

where D stands for the determinant

$$D = \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}\,. \tag{1315}$$

For the case under consideration

$$D = c(ab - h^2) = \phi^4(\phi^2 + \kappa_2\bar{\omega}^2)\,. \tag{1316}$$

Hence, according to equations (1312), (1314), and (1316), we have

$$\mathfrak{N} = \frac{\text{constant}}{\phi^2\sqrt{\phi^2 + \kappa_2\bar{\omega}^2}}\, e^{-\frac{1}{\phi^2}\left\{\left(q_1 - \frac{\beta^2\phi^2}{\phi^2+\kappa_2\bar{\omega}^2}\right)\bar{\omega}^2 + q_2 z^2\right\} - 2\mathfrak{V}^*}\,. \tag{1317}$$

This completes our discussion of case (A).

Case (B).—According to (1263), we can now write

$$\mathfrak{V}(\bar{\omega}, z, t) = \mathfrak{V}_1(\bar{\omega}, t) + \mathfrak{V}_2(z, t)\,, \tag{1318}$$

where $\mathfrak{V}_1$ and $\mathfrak{V}_2$ are arbitrary functions of the arguments specified. Further, with the coefficients of the velocity ellipsoid and the Δ's

given according to equations (1264) and (1265), the compatibility conditions we have to consider are

$$\left.\begin{aligned}
(\kappa_1 + \kappa_2 y^2)\frac{\partial \mathfrak{V}_1}{\partial x} - \kappa_2 xy \frac{\partial \mathfrak{V}_1}{\partial y} + \tfrac{1}{2} x \frac{d^2\kappa_1}{dt^2} &= -\frac{1}{2}\frac{\partial \chi}{\partial x}\,,\\
(\kappa_1 + \kappa_2 x^2)\frac{\partial \mathfrak{V}_1}{\partial y} - \kappa_2 xy \frac{\partial \mathfrak{V}_1}{\partial x} + \tfrac{1}{2} y \frac{d^2\kappa_1}{dt^2} &= -\frac{1}{2}\frac{\partial \chi}{\partial y}\,,\\
\kappa_3 \frac{\partial \mathfrak{V}_2}{\partial z} + \tfrac{1}{2} z \frac{d^2\kappa_3}{dt^2} &= -\frac{1}{2}\frac{\partial \chi}{\partial z}\,,\\
\left(\frac{1}{2}\frac{d\kappa_1}{dt}x + \beta y\right)\frac{\partial \mathfrak{V}_1}{\partial x} + \left(\frac{1}{2}\frac{d\kappa_1}{dt}y - \beta x\right)\frac{\partial \mathfrak{V}_1}{\partial y} &\\
+ \frac{1}{2}\frac{d\kappa_3}{dt} z \frac{\partial \mathfrak{V}_2}{\partial z} &= \frac{1}{2}\frac{\partial \chi}{\partial t}\,.
\end{aligned}\right\} \quad (1319)$$

The foregoing equations can be further simplified to take the forms (cf. Eqs. [1271], [1272], and [1275])

$$\kappa_1 \frac{\partial \mathfrak{V}_1}{\partial \tau} + \frac{1}{2}\frac{d^2\kappa_1}{dt^2} = -\frac{1}{2}\frac{\partial \chi}{\partial \tau}\,, \qquad (1320)$$

$$\kappa_3 \frac{\partial \mathfrak{V}_2}{\partial \zeta} + \frac{1}{2}\frac{d^2\kappa_3}{dt^2} = -\frac{1}{2}\frac{\partial \chi}{\partial \zeta}\,, \qquad (1321)$$

$$\tau \frac{d\kappa_1}{dt}\frac{\partial \mathfrak{V}_1}{\partial \tau} + \zeta \frac{d\kappa_3}{dt}\frac{\partial \mathfrak{V}_2}{\partial \zeta} = \frac{1}{2}\frac{\partial \chi}{\partial t}\,. \qquad (1322)$$

The integrability condition resulting from equations (1320) and (1321) is clearly satisfied. On the other hand, from equations (1320) and (1322) we obtain

$$\kappa_1 \frac{\partial^2 \mathfrak{V}_1}{\partial t \partial \tau} + \tau \frac{d\kappa_1}{dt}\frac{\partial^2 \mathfrak{V}_1}{\partial \tau^2} + 2\frac{d\kappa_1}{dt}\frac{\partial \mathfrak{V}_1}{\partial \tau} + \frac{1}{2}\frac{d^3\kappa_1}{dt^3} = 0\,. \qquad (1323)$$

Similarly, from equations (1321) and (1322) we find that

$$\kappa_3 \frac{\partial^2 \mathfrak{V}_2}{\partial t \partial \zeta} + \zeta \frac{d\kappa_3}{dt}\frac{\partial^2 \mathfrak{V}_2}{\partial \zeta^2} + 2\frac{d\kappa_3}{dt}\frac{\partial \mathfrak{V}_2}{\partial \zeta} + \frac{1}{2}\frac{d^3\kappa_3}{dt^3} = 0\,. \qquad (1324)$$

Equations (1323) and (1324) are identical in forms with the equations (1293) and (1294). We can therefore write (cf. Eqs. [1295])

$$\left.\begin{aligned}\frac{\partial \mathfrak{V}_1}{\partial \tau} &= \left(\frac{q_1}{\phi_1^4} - \frac{\ddot{\phi}_1}{\phi_1}\right) + \frac{1}{\phi_1^4} F_1\left(\frac{\tau}{\phi_1^2}\right),\\ \frac{\partial \mathfrak{V}_2}{\partial \zeta} &= \left(\frac{q_2}{\phi_2^4} - \frac{\ddot{\phi}_2}{\phi_2}\right) + \frac{1}{\phi_2^4} F_2\left(\frac{\zeta}{\phi_2^2}\right),\end{aligned}\right\} \tag{1325}$$

where

$$\kappa_1 = \phi_1^2 \; ; \qquad \kappa_3 = \phi_2^2 \tag{1326}$$

and where F_1 and F_2 are arbitrary functions of the arguments specified and q_1 and q_2 are constants. According to equations (1318) and (1325), we can now write

$$\left.\begin{aligned}\mathfrak{V} = \left(\frac{q_1}{\phi_1^4} - \frac{\ddot{\phi}_1}{\phi_1}\right)\tau + \left(\frac{q_2}{\phi_2^4} - \frac{\ddot{\phi}_2}{\phi_2}\right)\zeta + \frac{1}{\phi_1^2}\mathfrak{V}_1^*\left(\frac{\tau}{\phi_1^2}\right)&\\ + \frac{1}{\phi_2^2}\mathfrak{V}_2^*\left(\frac{\zeta}{\phi_2^2}\right) + \mathfrak{V}_0(t)\,,&\end{aligned}\right\} \tag{1327}$$

where $\mathfrak{V}_1^*$, $\mathfrak{V}_2^*$, and $\mathfrak{V}_0$ are arbitrary functions of the arguments specified. The solutions for χ and σ are obtained by methods similar to those used in our discussion of case (A) above. We find (cf. Eqs. [1303] and [1312])

$$\left.\begin{aligned}-\tfrac{1}{2}\chi = \left(\frac{q_1}{\phi_1^2} + \dot{\phi}_1^2\right)\tau + \left(\frac{q_2}{\phi_2^2} + \dot{\phi}_2^2\right)\zeta + \mathfrak{V}_1^*\left(\frac{\tau}{\phi_1^2}\right)&\\ + \mathfrak{V}_2^*\left(\frac{\zeta}{\phi_2^2}\right) + \text{constant}&\end{aligned}\right\} \tag{1328}$$

and

$$\left.\begin{aligned}\sigma = \frac{1}{\phi_1^2}\left(q_1 - \frac{\beta^2\phi_1^2}{\phi_1^2 + \kappa_2\bar{\omega}^2}\right)\bar{\omega}^2 + \frac{q_2}{\phi_2^2}z^2 + 2\mathfrak{V}_1^*\left(\frac{\tau}{\phi_1^2}\right)&\\ + 2\mathfrak{V}_2^*\left(\frac{\zeta}{\phi_2^2}\right) + \text{constant}\,.&\end{aligned}\right\} \tag{1329}$$

Further, we have (cf. Eqs. [1306] and [1307])

$$\left.\begin{aligned} U_0 &= \frac{\dot{\phi}_1}{\phi_1}\, x + \frac{\beta y}{\phi_1^2 + \kappa_2(x^2 + y^2)}\,, \\ V_0 &= \frac{\dot{\phi}_1}{\phi_1}\, y - \frac{\beta x}{\phi_1^2 + \kappa_2(x^2 + y^2)}\,, \\ W_0 &= \frac{\dot{\phi}_2}{\phi_2}\, z\,, \end{aligned}\right\} \tag{1330}$$

and

$$\Pi_0 = \frac{\dot{\phi}_1}{\phi_1}\,\bar{\omega}\,; \qquad \Theta_0 = \frac{-\beta\bar{\omega}}{\phi_1^2 + \kappa_2\bar{\omega}^2}\,; \qquad Z_0 = \frac{\dot{\phi}_2}{\phi_2}\, z\,. \tag{1331}$$

Finally, for a distribution function of the form (1313) we have

$$\mathfrak{N} = \frac{\text{constant}}{\phi_1\phi_2\sqrt{\phi_1^2 + \kappa_2\bar{\omega}^2}}\, e^{-\left[\frac{1}{\phi_1^2}\left(q_1 - \frac{\beta^2\phi_1^2}{\phi_1^2+\kappa_2\bar{\omega}^2}\right)\bar{\omega}^2 + \frac{q_2}{\phi_2^2}z^2 + 2\mathfrak{B}_1^* + 2\mathfrak{B}_2^*\right]}\,. \tag{1332}$$

The discussion of the further consequences of the foregoing relations are postponed to a future occasion.

60. *Some remarks on nonhomogeneous linear partial differential equations.*—In § 59 we encountered the nonhomogeneous equation (Eq. [1281])

$$\left.\begin{aligned} \frac{\partial}{\partial t}(\kappa_1\mathfrak{B}) + \frac{1}{\kappa_1}\frac{d\kappa_1}{dt}\left\{\tau\frac{\partial}{\partial\tau}(\kappa_1\mathfrak{B}) + \zeta\frac{\partial}{\partial\zeta}(\kappa_1\mathfrak{B})\right\}& \\ = -\tfrac{1}{2}(\tau + \zeta)\frac{d^3\kappa_1}{dt^3} + F^*(t)& \end{aligned}\right\} \tag{1333}$$

and found that the most general solution of this equation is

$$\mathfrak{B} = \left(\frac{q_1}{\phi^4} - \frac{\ddot{\phi}}{\phi}\right)\tau + \left(\frac{q_2}{\phi^4} - \frac{\ddot{\phi}}{\phi}\right)\zeta + \frac{1}{\phi^2}\,\mathfrak{B}^*\left(\frac{\tau}{\phi^2}, \frac{\zeta}{\phi^2}\right) + \mathfrak{B}_0(t)\,, \tag{1334}$$

where $\kappa_1 = \phi^2$ and $\mathfrak{B}^*$ and $\mathfrak{B}_0$ are arbitrary functions of the arguments specified. Since

$$\frac{q_1}{\phi^4}\,\tau + \frac{q_2}{\phi^4}\,\zeta \tag{1335}$$

can be absorbed into $\mathfrak{B}^*/\phi^2$, we can express the solution (1334) also in the form

$$\mathfrak{B} = -\frac{\ddot{\phi}}{\phi}(\tau + \zeta) + \frac{1}{\phi^2}\mathfrak{B}^*\left(\frac{\tau}{\phi^2}, \frac{\zeta}{\phi^2}\right) + \mathfrak{B}_0(t) . \qquad (1336)$$

We shall now make some general remarks on this solution of the nonhomogeneous equation (1333).

First, we may notice that there is no loss of generality if we set $F^* = 0$ in (1333). For, if we let

$$\mathfrak{B} = \mathfrak{B}_1(\bar{\omega}, z, t) + \mathfrak{B}_0(t) , \qquad (1337)$$

where

$$\mathfrak{B}_0(t) = \frac{1}{\kappa_1}\int^t F^*(t)dt , \qquad (1338)$$

then the substitution of (1337) in (1333) leads to the differential equation

$$\left.\begin{aligned}\frac{\partial}{\partial t}(\kappa_1\mathfrak{B}_1) + \frac{1}{\kappa_1}\frac{d\kappa_1}{dt}\left\{\tau\frac{\partial}{\partial\tau}(\kappa_1\mathfrak{B}_1) + \zeta\frac{\partial}{\partial\zeta}(\kappa_1\mathfrak{B}_1)\right\}& \\ = -\tfrac{1}{2}(\tau + \zeta)\frac{d^3\kappa_1}{dt^3}&\end{aligned}\right\} \qquad (1339)$$

for $\mathfrak{B}_1$.

Now, the *homogeneous equation* associated with (1339) is

$$\frac{\partial}{\partial t}(\kappa_1\mathfrak{B}_1) + \frac{1}{\kappa_1}\frac{d\kappa_1}{dt}\left\{\tau\frac{\partial}{\partial\tau}(\kappa_1\mathfrak{B}_1) + \zeta\frac{\partial}{\partial\zeta}(\kappa_1\mathfrak{B}_1)\right\} = 0 . \qquad (1340)$$

The equations for the characteristics of (1340) are

$$dt = \frac{d\tau}{\dfrac{\tau}{\kappa_1}\dfrac{d\kappa_1}{dt}} = \frac{d\zeta}{\dfrac{\zeta}{\kappa_1}\dfrac{d\kappa_1}{dt}} . \qquad (1341)$$

The foregoing equations are the same as (1287). Two independent integrals of (1341) are (cf. Eq. [1289])

$$\frac{\tau}{\kappa_1} = \text{constant} ; \qquad \frac{\zeta}{\kappa_1} = \text{constant} . \qquad (1342)$$

Hence,

$$\kappa_1 \mathfrak{V}_1 = \mathfrak{V}^* \left(\frac{\tau}{\kappa_1}, \frac{\zeta}{\kappa_1} \right), \tag{1343}$$

where $\mathfrak{V}^*$ is an arbitrary function of the arguments specified. To find the general solution of (1339), we must add to the solution (1343) of the homogeneous equation the *most general particular integral.* However, physical considerations might often suggest particular integrals of certain special forms though they may not correspond to the most general particular integral. We shall later encounter such circumstances (cf. § 73), but meantime we may notice how we can discover particular integrals of differential equations like (1339):

Try a particular integral of the form

$$\mathfrak{V}_1 = C_1(t)\tau + C_2(t)\zeta, \tag{1344}$$

where C_1 and C_2 are functions of time only. Substituting (1344) in (1339), we find that

$$\left.\begin{aligned} \tau \left\{ \frac{d}{dt}(\kappa_1 C_1) + C_1 \frac{d\kappa_1}{dt} + \frac{1}{2}\frac{d^3\kappa_1}{dt^3} \right\} \\ + \zeta \left\{ \frac{d}{dt}(\kappa_1 C_2) + C_2 \frac{d\kappa_1}{dt} + \frac{1}{2}\frac{d^3\kappa_1}{dt^3} \right\} = 0 ; \end{aligned}\right\} \tag{1345}$$

or, equating the coefficients of τ and ζ, we obtain

$$\left.\begin{aligned} \kappa_1 \frac{dC_1}{dt} + 2C_1 \frac{d\kappa_1}{dt} + \frac{1}{2}\frac{d^3\kappa_1}{dt^3} = 0, \\ \kappa_1 \frac{dC_2}{dt} + 2C_2 \frac{d\kappa_1}{dt} + \frac{1}{2}\frac{d^3\kappa_1}{dt^3} = 0. \end{aligned}\right\} \tag{1346}$$

We have already encountered in Part X an equation identical in form with the foregoing equations (cf. Eq. [1035]). We can therefore write (cf. Eq. [1045])

$$C_1 = \left(\frac{q_1}{\phi^4} - \frac{\ddot{\phi}}{\phi} \right); \qquad C_2 = \left(\frac{q_2}{\phi^4} - \frac{\ddot{\phi}}{\phi} \right), \tag{1347}$$

where $\kappa_1 = \phi^2$ and q_1 and q_2 are constants. Adding the particular integral thus obtained to the general solution of the homogeneous equation given by (1343), we see that we have recovered the gen-

eral solution (1334). However, this "straightforward" method has the disadvantage that we do not know if the solution obtained in this manner is, in fact, the most general solution.[52]

XII. THE EVOLUTION OF ELLIPSOIDAL SYSTEMS

61. *The statement of the problem.*—In Part VIII we considered the dynamics of ellipsoidal stellar systems under steady-state conditions. We shall now consider the analogous problem under nonsteady-state conditions. More particularly, we shall be concerned with the motions inside a homogeneous ellipsoidal distribution of mass. In discussing this problem under nonsteady-state circumstances we naturally regard the density and the ratio of the axes of the homogeneous ellipsoid as functions of the time, arbitrary in the first instance. In other words, the gravitational potential, $\mathfrak{V}$, is assumed to have the

$$\mathfrak{V} = \tfrac{1}{2}[\alpha_1(t)x^2 + \alpha_2(t)y^2 + \alpha_3(t)z^2] + \mathfrak{V}_0(t) , \qquad (1348)$$

form where α_1, α_2, α_3, and $\mathfrak{V}_0$ are functions of time only.

The analysis of this problem, which now follows, is divided into five main sections:

In § 62 we obtain the fundamental equations of the problem: it is found that we are led to consider a system of seventy equations. In § 63 we prove certain theorems relating to the solutions of a certain type of differential equation which plays a fundamental role in the further developments of the theory. In § 64 spherical systems are considered; § 65 is devoted to spheroidal systems; §§ 66 and 67, to a further consideration of these systems and certain applications to the evolution of elliptical nebulae; and, finally, § 68 contains a preliminary discussion of general ellipsoidal systems.

62. *The equations of the problem.*—The mathematical problem hinges on the relations (cf. Eqs. [956] and [957])

$$\left.\begin{aligned} \boldsymbol{A}\,\mathrm{grad}\,\mathfrak{V} + \frac{\partial\boldsymbol{\Delta}}{\partial t} &= -\tfrac{1}{2}\,\mathrm{grad}\,\chi\,, \\ \boldsymbol{\Delta}\cdot\mathrm{grad}\,\mathfrak{V} &= \frac{1}{2}\frac{\partial\chi}{\partial t}\,, \end{aligned}\right\} \qquad (1349)$$

[52] These remarks, while they may appear irrelevant in the present connection, have considerable importance when we come to discuss certain other types of nonhomogeneous linear differential equations which are more complicated than the one we have been specifically dealing with in § 60 (cf. Part XIII, § 73).

where the elements of the matrix A and the components of the vector $\boldsymbol{\Delta}$ are given by equations (941) and (953).[53]

For the case under consideration

$$\operatorname{grad} \mathfrak{B} = (\alpha_1 x,\ \alpha_2 y,\ \alpha_3 z) \,. \tag{1350}$$

Combining equations (1349) and (1350), we have

$$\alpha_1 a x + \alpha_2 h y + \alpha_3 g z + \frac{\partial \Delta_1}{\partial t} = -\frac{1}{2}\frac{\partial \chi}{\partial x} \,, \tag{1351}$$

$$\alpha_1 h x + \alpha_2 b y + \alpha_3 f z + \frac{\partial \Delta_2}{\partial t} = -\frac{1}{2}\frac{\partial \chi}{\partial y} \,, \tag{1352}$$

$$\alpha_1 g x + \alpha_2 f y + \alpha_3 c z + \frac{\partial \Delta_3}{\partial t} = -\frac{1}{2}\frac{\partial \chi}{\partial z} \,, \tag{1353}$$

and

$$\alpha_1 \Delta_1 x + \alpha_2 \Delta_2 y + \alpha_3 \Delta_3 z = \frac{1}{2}\frac{\partial \chi}{\partial t} \,. \tag{1354}$$

The foregoing equations lead to six integrability conditions. Thus, from equations (1351) and (1352) we obtain (cf. Eq. [877])

$$\left.\begin{aligned} 3\alpha_1 x \frac{\partial h}{\partial x} - 3\alpha_2 y \frac{\partial h}{\partial y} + \alpha_3 z\left(\frac{\partial f}{\partial x} - \frac{\partial g}{\partial y}\right) + (\alpha_1 - \alpha_2) h & \\ + \frac{\partial}{\partial t}\left(\frac{\partial \Delta_2}{\partial x} - \frac{\partial \Delta_1}{\partial y}\right) = 0 \,. & \end{aligned}\right\} \tag{1355}$$

The two further integrability conditions which arise from the equations (1351)–(1353) can be obtained from the foregoing equation by cyclically permuting (x, y, z), $(\alpha_1, \alpha_2, \alpha_3)$, $(\Delta_1, \Delta_2, \Delta_3)$, and (f, g, h).

Similarly, from equations (1351) and (1354) we find (after some reductions) that

$$\left.\begin{aligned} 3\alpha_1 x \frac{\partial \Delta_1}{\partial x} + \alpha_2 y\left(2\frac{\partial \Delta_2}{\partial x} + \frac{\partial \Delta_1}{\partial y}\right) + \alpha_3 z\left(2\frac{\partial \Delta_3}{\partial x} + \frac{\partial \Delta_1}{\partial z}\right) & \\ + a x \frac{d\alpha_1}{dt} + h y \frac{d\alpha_2}{dt} + g z \frac{d\alpha_3}{dt} + \frac{\partial^2 \Delta_1}{\partial t^2} + \alpha_1 \Delta_1 = 0 \,. & \end{aligned}\right\} \tag{1356}$$

[53] Some relations between the coefficients introduced in these solutions are given in Eqs. (942) and (949).

There are two similar equations which can be obtained from (1356) by appropriately permuting cyclically the various quantities in this equation.

Substituting for the coefficients of the velocity ellipsoid and for the Δ's according to equations (941) and (953) in (1355) and (1356), we obtain

$$\left.\begin{aligned}
&3\alpha_1 x(h_{20} + h_{21}z + h_{40}y) - 3\alpha_2 y(h_{30} + f_{21}z + h_{40}x)\\
&\quad + \alpha_3 z(f_{11} - g_{11} - 3h_{21}x + 3f_{21}y) + (\alpha_1 - \alpha_2)[h_{10} + h_{11}z\\
&\quad - g_{21}z^2 + (h_{20} + h_{21}z)x + (h_{30} + f_{21}z)y + h_{40}xy]\\
&\quad + \frac{\partial}{\partial t}\left[3x\frac{dh_{20}}{dt} - 3y\frac{dh_{30}}{dt} + z\frac{d}{dt}(f_{11} - g_{11}) + \gamma_3 - \beta_3\right] = 0
\end{aligned}\right\} \quad (1357)$$

and

$$\left.\begin{aligned}
&3xy(\alpha_2 - \alpha_1)\frac{dh_{20}}{dt} + 3zx(\alpha_3 - \alpha_1)\frac{dg_{30}}{dt} - \tfrac{3}{2}\alpha_1 x\frac{da_0}{dt} + \alpha_2 y(\beta_3 + 2\gamma_3)\\
&\quad + \alpha_3 z(\gamma_2 + 2\beta_2) - yz\left[(\alpha_2 + \alpha_3)\frac{df_{11}}{dt} + 2\alpha_2\frac{dg_{11}}{dt} + 2\alpha_3\frac{dh_{11}}{dt}\right]\\
&\quad - \frac{d\alpha_1}{dt}(2h_{20}xy + 2h_{21}xyz + h_{40}xy^2 + a_0 x + 2g_{30}xz + g_{40}z^2x)\\
&\quad + \frac{d\alpha_2}{dt}(h_{10}y + h_{11}yz - g_{21}z^2y + h_{20}xy + h_{21}xyz + h_{30}y^2 + f_{21}y^2z + h_{40}xy^2)\\
&\quad + \frac{d\alpha_3}{dt}(g_{10}z + g_{11}yz - f_{21}y^2z + g_{20}z^2 + g_{21}yz^2 + g_{30}xz + h_{21}xyz + g_{40}z^2x)\\
&\quad + y^2\frac{d^3h_{30}}{dt^3} - xy\frac{d^3h_{20}}{dt^3} - xz\frac{d^3g_{30}}{dt^3} + z^2\frac{d^3g_{20}}{dt^3} - yz\frac{d^3f_{11}}{dt^3} - \frac{1}{2}\frac{d^3a_0}{dt^3}x\\
&\quad + \frac{d^2\beta_3}{dt^2}y + \frac{d^2\gamma_2}{dt^2}z + \frac{d^2\delta_1}{dt^2} + \alpha_1\left(y^2\frac{dh_{30}}{dt} - xy\frac{dh_{20}}{dt} - xz\frac{dg_{30}}{dt}\right.\\
&\quad \left. + z^2\frac{dg_{20}}{dt} - yz\frac{df_{11}}{dt} - \frac{1}{2}\frac{da_0}{dt}x + \beta_3 y + \gamma_2 z + \delta_1\right) = 0\,.
\end{aligned}\right\} \quad (1358)$$

Equating, now, the coefficients of the different powers and the different combinations of powers of x, y, and z in the two foregoing equations and combining them with the equations which result from the two other pairs of equations (similar to the Eqs. [1357] and [1358]), we obtain a set of sixty-six equations. These sixty-six equa-

tions, when appropriately grouped together, give rise to the following fourteen systems of equations:[54]

$$\left.\begin{array}{lll}
f_{21}(\alpha_3-\alpha_1)=0\,; & g_{21}(\alpha_1-\alpha_2)=0\,; & h_{21}(\alpha_2-\alpha_3)=0\,,\\[4pt]
f_{21}\dfrac{d}{dt}(\alpha_2-\alpha_3)=0\,; & g_{21}\dfrac{d}{dt}(\alpha_3-\alpha_1)=0\,; & h_{21}\dfrac{d}{dt}(\alpha_1-\alpha_2)=0\,,\\[4pt]
f_{21}\dfrac{d}{dt}(\alpha_2-\alpha_1)=0\,; & g_{21}\dfrac{d}{dt}(\alpha_3-\alpha_2)=0\,; & h_{21}\dfrac{d}{dt}(\alpha_1-\alpha_3)=0\,,\\[4pt]
f_{21}(\alpha_1-4\alpha_2+3\alpha_3)=0\,; & g_{21}(\alpha_2-4\alpha_3+3\alpha_1)=0\,; & h_{21}(\alpha_3-4\alpha_1+3\alpha_2)=0\,,\\[4pt]
f_{21}(4\alpha_2-\alpha_3-3\alpha_1)=0\,; & g_{21}(4\alpha_3-\alpha_1-3\alpha_2)=0\,; & h_{21}(4\alpha_1-\alpha_2-3\alpha_3)=0\,,\\[4pt]
f_{21}\dfrac{d}{dt}(\alpha_1-2\alpha_2+\alpha_3)=0\,; & g_{21}\dfrac{d}{dt}(\alpha_2-2\alpha_3+\alpha_1)=0\,; & h_{21}\dfrac{d}{dt}(\alpha_3-2\alpha_1+\alpha_2)=0\,,
\end{array}\right\}\ \mathrm{I}$$

$$\left.\begin{array}{lll}
f_{40}(\alpha_2-\alpha_3)=0\,; & g_{40}(\alpha_3-\alpha_1)=0\,; & h_{40}(\alpha_1-\alpha_2)=0\,,\\[4pt]
f_{40}\dfrac{d}{dt}(\alpha_3-\alpha_2)=0\,; & g_{40}\dfrac{d}{dt}(\alpha_1-\alpha_3)=0\,; & h_{40}\dfrac{d}{dt}(\alpha_2-\alpha_1)=0\,,\\[4pt]
f_{40}\dfrac{d}{dt}(\alpha_2-\alpha_3)=0\,; & g_{40}\dfrac{d}{dt}(\alpha_3-\alpha_1)=0\,; & h_{40}\dfrac{d}{dt}(\alpha_1-\alpha_2)=0\,,
\end{array}\right\}\ \mathrm{II}$$

$$\left.\begin{array}{lr}
f_{11}+g_{11}+h_{11}=0\,, & \text{(i)}\\[6pt]
\left.\begin{aligned}
&\frac{d^2}{dt^2}(f_{11}-g_{11})+\alpha_3(f_{11}-g_{11})+(\alpha_1-\alpha_2)h_{11}=0\,,\\
&\frac{d^2}{dt^2}(g_{11}-h_{11})+\alpha_1(g_{11}-h_{11})+(\alpha_2-\alpha_3)f_{11}=0\,,\\
&\frac{d^2}{dt^2}(h_{11}-f_{11})+\alpha_2(h_{11}-f_{11})+(\alpha_3-\alpha_1)g_{11}=0\,,
\end{aligned}\right\} & \text{(ii)}\\[6pt]
\left.\begin{aligned}
&\frac{d^3f_{11}}{dt^3}+(\alpha_1+\alpha_2+\alpha_3)\frac{df_{11}}{dt}+2\alpha_2\frac{dg_{11}}{dt}+2\alpha_3\frac{dh_{11}}{dt}-h_{11}\frac{d\alpha_2}{dt}-g_{11}\frac{d\alpha_3}{dt}=0\,,\\
&\frac{d^3g_{11}}{dt^3}+(\alpha_1+\alpha_2+\alpha_3)\frac{dg_{11}}{dt}+2\alpha_3\frac{dh_{11}}{dt}+2\alpha_1\frac{df_{11}}{dt}-f_{11}\frac{d\alpha_3}{dt}-h_{11}\frac{d\alpha_1}{dt}=0\,,\\
&\frac{d^3h_{11}}{dt^3}+(\alpha_1+\alpha_2+\alpha_3)\frac{dh_{11}}{dt}+2\alpha_1\frac{df_{11}}{dt}+2\alpha_2\frac{dg_{11}}{dt}-g_{11}\frac{d\alpha_1}{dt}-f_{11}\frac{d\alpha_2}{dt}=0\,,
\end{aligned}\right\} & \text{(iii)}
\end{array}\right\}\ \mathrm{III}$$

[54] The fourteen systems of equations which now follow include the sixty-six relations already mentioned and the four other relations given by Eqs. (942) and (949). We have thus to deal with a system of seventy equations.

$$\left.\begin{aligned}
3\,\frac{d^2f_{20}}{dt^2} + (4\alpha_2 - \alpha_3)f_{20} &= 0\,, && \text{(i)}\\
\frac{d^3f_{20}}{dt^3} + \alpha_3\,\frac{df_{20}}{dt} + f_{20}\,\frac{d\alpha_2}{dt} &= 0\,, && \text{(ii)}\\
-\frac{d^3f_{20}}{dt^3} + (3\alpha_3 - 4\alpha_2)\,\frac{df_{20}}{dt} + f_{20}\,\frac{d}{dt}\,(\alpha_3 - 2\alpha_2) &= 0\,, && \text{(iii)}
\end{aligned}\right\}\ \text{IV}$$

$$\left.\begin{aligned}
3\,\frac{d^2g_{20}}{dt^2} + (4\alpha_3 - \alpha_1)g_{20} &= 0\,, && \text{(i)}\\
\frac{d^3g_{20}}{dt^3} + \alpha_1\,\frac{dg_{20}}{dt} + g_{20}\,\frac{d\alpha_3}{dt} &= 0\,, && \text{(ii)}\\
-\frac{d^3g_{20}}{dt^3} + (3\alpha_1 - 4\alpha_3)\,\frac{dg_{20}}{dt} + g_{20}\,\frac{d}{dt}\,(\alpha_1 - 2\alpha_3) &= 0\,, && \text{(iii)}
\end{aligned}\right\}\ \text{V}$$

$$\left.\begin{aligned}
3\,\frac{d^2h_{20}}{dt^2} + (4\alpha_1 - \alpha_2)h_{20} &= 0\,, && \text{(i)}\\
\frac{d^3h_{20}}{dt^3} + \alpha_2\,\frac{dh_{20}}{dt} + h_{20}\,\frac{d\alpha_1}{dt} &= 0\,, && \text{(ii)}\\
-\frac{d^3h_{20}}{dt^3} + (3\alpha_2 - 4\alpha_1)\,\frac{dh_{20}}{dt} + h_{20}\,\frac{d}{dt}\,(\alpha_2 - 2\alpha_1) &= 0\,, && \text{(iii)}
\end{aligned}\right\}\ \text{VI}$$

$$\left.\begin{aligned}
3\,\frac{d^2f_{30}}{dt^2} + (4\alpha_3 - \alpha_2)f_{30} &= 0\,, && \text{(i)}\\
\frac{d^3f_{20}}{dt^3} + \alpha_2\,\frac{df_{30}}{dt} + f_{30}\,\frac{d\alpha_3}{dt} &= 0\,, && \text{(ii)}\\
-\frac{d^3f_{30}}{dt^3} + (3\alpha_2 - 4\alpha_3)\,\frac{df_{30}}{dt} + f_{30}\,\frac{d}{dt}\,(\alpha_2 - 2\alpha_3) &= 0\,, && \text{(iii)}
\end{aligned}\right\}\ \text{VII}$$

$$\left.\begin{aligned}
3\,\frac{d^2g_{30}}{dt^2} + (4\alpha_1 - \alpha_3)g_{30} &= 0\,, && \text{(i)}\\
\frac{d^3g_{30}}{dt^3} + \alpha_3\,\frac{dg_{30}}{dt} + g_{30}\,\frac{d\alpha_1}{dt} &= 0\,, && \text{(ii)}\\
-\frac{d^3g_{30}}{dt^3} + (3\alpha_3 - 4\alpha_1)\,\frac{dg_{30}}{dt} + g_{30}\,\frac{d}{dt}\,(\alpha_3 - 2\alpha_1) &= 0\,, && \text{(iii)}
\end{aligned}\right\}\ \text{VIII}$$

$$\left.\begin{aligned}
&3\frac{d^3h_{30}}{dt^3} + (4\alpha_2 - \alpha_1)h_{30} = 0\,, && \text{(i)}\\
&\frac{d^3h_{30}}{dt^3} + \alpha_1\frac{dh_{30}}{dt} + h_{30}\frac{d\alpha_2}{dt} = 0\,, && \text{(ii)}\\
&-\frac{d^3h_{30}}{dt^3} + (3\alpha_1 - 4\alpha_2)\frac{dh_{30}}{dt} + h_{30}\frac{d}{dt}(\alpha_1 - 2\alpha_2) = 0\,, && \text{(iii)}
\end{aligned}\right\}\ \text{IX}$$

$$\left.\begin{aligned}
&\frac{df_{10}}{dt} = \beta_1 + \gamma_1\,, && \text{(i)}\\
&\frac{d}{dt}(\beta_1 - \gamma_1) = (\alpha_2 - \alpha_3)f_{10}\,, && \text{(ii)}\\
&\frac{d^2\beta_1}{dt^2} + \alpha_2\beta_1 + \alpha_3(\beta_1 + 2\gamma_1) + f_{10}\frac{d\alpha_3}{dt} = 0\,, && \text{(iii)}\\
&\frac{d^2\gamma_1}{dt^2} + \alpha_3\gamma_1 + \alpha_2(\gamma_1 + 2\beta_1) + f_{10}\frac{d\alpha_2}{dt} = 0\,, && \text{(iv)}
\end{aligned}\right\}\ \text{X}$$

$$\left.\begin{aligned}
&\frac{dg_{10}}{dt} = \beta_2 + \gamma_2\,, && \text{(i)}\\
&\frac{d}{dt}(\beta_2 - \gamma_2) = (\alpha_3 - \alpha_1)g_{10}\,, && \text{(ii)}\\
&\frac{d^2\beta_2}{dt^2} + \alpha_3\beta_2 + \alpha_1(\beta_2 + 2\gamma_2) + g_{10}\frac{d\alpha_1}{dt} = 0\,, && \text{(iii)}\\
&\frac{d^2\gamma_2}{dt^2} + \alpha_1\gamma_2 + \alpha_3(\gamma_2 + 2\beta_2) + g_{10}\frac{d\alpha_3}{dt} = 0\,, && \text{(iv)}
\end{aligned}\right\}\ \text{XI}$$

$$\left.\begin{aligned}
&\frac{dh_{10}}{dt} = \beta_3 + \gamma_3\,, && \text{(i)}\\
&\frac{d}{dt}(\beta_3 - \gamma_3) = (\alpha_1 - \alpha_2)h_{10}\,, && \text{(ii)}\\
&\frac{d^2\beta_3}{dt^2} + \alpha_1\beta_3 + \alpha_2(\beta_3 + 2\gamma_3) + h_{10}\frac{d\alpha_2}{dt} = 0\,, && \text{(iii)}\\
&\frac{d^2\gamma_3}{dt^2} + \alpha_2\gamma_3 + \alpha_1(\gamma_3 + 2\beta_3) + h_{10}\frac{d\alpha_1}{dt} = 0\,, && \text{(iv)}
\end{aligned}\right\}\ \text{XII}$$

$$\left.\begin{aligned} a_0\frac{d\alpha_1}{dt}+2\alpha_1\frac{da_0}{dt}+\frac{1}{2}\frac{d^3a_0}{dt^3}&=0\,,\\ b_0\frac{d\alpha_2}{dt}+2\alpha_2\frac{db_0}{dt}+\frac{1}{2}\frac{d^3b_0}{dt^3}&=0\,,\\ c_0\frac{d\alpha_3}{dt}+2\alpha_3\frac{dc_0}{dt}+\frac{1}{2}\frac{d^3c_0}{dt^3}&=0\,,\end{aligned}\right\} \qquad \text{XIII}$$

and

$$\frac{d^2\delta_1}{dt^2}+\alpha_1\delta_1=0\,;\qquad \frac{d^2\delta_2}{dt^2}+\alpha_2\delta_2=0\,;\qquad \frac{d^2\delta_3}{dt^2}+\alpha_3\delta_3=0\,. \quad \text{XIV}$$

The discussion of the foregoing systems of equations are undertaken in §§ 64–68.

63. *Theorems concerning the solutions of a certain type of differential equation.*—Consider the differential equations

$$\alpha=\frac{q_i}{\phi_i^4}-\frac{\ddot{\phi}_i}{\phi_i} \qquad (1359)$$

and

$$\alpha=\frac{q_j}{\phi_j^4}-\frac{\ddot{\phi}_j}{\phi_j}\,, \qquad (1360)$$

where α, ϕ_i, and ϕ_j are functions of t and q_i and q_j are two arbitrary real constants. We shall show that any solution of (1359) can be expressed in terms of any solution of (1360), and conversely:

According to the equations (1359) and (1360), we can write

$$\ddot{\phi}_i\phi_j-\ddot{\phi}_j\phi_i=\phi_i\phi_j\left(\frac{q_i}{\phi_i^4}-\frac{q_j}{\phi_j^4}\right) \qquad (1361)$$

or differently as

$$\frac{d}{dt}(\dot{\phi}_i\phi_j-\dot{\phi}_j\phi_i)=\phi_i\phi_j\left(\frac{q_i}{\phi_i^4}-\frac{q_j}{\phi_j^4}\right). \qquad (1362)$$

Let

$$\phi_i=u_{ij}\phi_j\,. \qquad (1363)$$

Then

$$\dot{\phi}_i\phi_j - \dot{\phi}_j\phi_i = \phi_j^2\dot{u}_{ij}\,, \tag{1364}$$

and equation (1362) can be re-written as

$$\frac{d}{dt}(\phi_j^2\dot{u}_{ij}) = \frac{u_{ij}}{\phi_j^2}\left(\frac{q_i}{u_{ij}^4} - q_j\right). \tag{1365}$$

The foregoing equation admits of an immediate first integral. For, multiplying both sides of (1365) by $\phi_j^2\dot{u}_{ij}$, we obtain

$$\frac{1}{2}\frac{d}{dt}(\phi_j^2\dot{u}_{ij})^2 = \left(\frac{q_i}{u_{ij}^3} - q_j u_{ij}\right)\dot{u}_{ij}\,, \tag{1366}$$

or, integrating, we have

$$(\phi_j^2\dot{u}_{ij})^2 = q_{ij} - \frac{q_i}{u_{ij}^2} - q_j u_{ij}^2\,, \tag{1367}$$

where q_{ij} is a constant of integration. Equation (1367) can be written alternatively as

$$\frac{1}{\phi_j^2} = \pm\frac{\dot{u}_{ij}}{\sqrt{q_{ij} - \dfrac{q_i}{u_{ij}^2} - q_j u_{ij}^2}}\,. \tag{1368}$$

Integrating the foregoing equation, we have

$$\int^t \frac{dt}{\phi_j^2} = \pm\int\frac{u_{ij}du_{ij}}{\sqrt{-\,q_i + q_{ij}u_{ij}^2 - q_j u_{ij}^4}}\,. \tag{1369}$$

Similarly, if

$$\phi_j = u_{ji}\phi_i\,, \tag{1370}$$

we have

$$\int^t \frac{dt}{\phi_i^2} = \mp\int\frac{u_{ji}du_{ji}}{\sqrt{-\,q_j + q_{ji}u_{ji}^2 - q_i u_{ji}^4}}\,, \tag{1371}$$

where, as may be readily verified, $q_{ij} = q_{ji}$.

The integrals occurring in the equations (1369) and (1371) are readily evaluated. We have

$$\left.\begin{aligned}
\pm 2\int^{t}\frac{dt}{\phi_j^2} &= \frac{1}{\sqrt{-q_j}}\sinh^{-1}\frac{q_{ij}-2q_ju_{ij}^2}{\sqrt{4q_iq_j-q_{ij}^2}} && (q_j<0\,;\quad 4q_iq_j>q_{ij}^2)\,,\\
&= \frac{1}{\sqrt{-q_j}}\cosh^{-1}\frac{q_{ij}-2q_ju_{ij}^2}{\sqrt{q_{ij}^2-4q_iq_j}} && (q_j<0\,;\quad q_{ij}^2>4q_iq_j)\,,\\
&= \frac{1}{\sqrt{q_j}}\cos^{-1}\frac{q_{ij}-2q_ju_{ij}^2}{\sqrt{q_{ij}^2-4q_iq_j}} && (q_j>0\,;\quad q_{ij}^2>4q_iq_j)\,.
\end{aligned}\right\}\quad(1372)$$

Similarly, from (1371) we obtain

$$\left.\begin{aligned}
\mp 2\int^{t}\frac{dt}{\phi_i^2} &= \frac{1}{\sqrt{-q_i}}\sinh^{-1}\frac{q_{ji}-2q_iu_{ji}^2}{\sqrt{4q_iq_j-q_{ji}^2}} && (q_i<0\,;\quad 4q_iq_j>q_{ji}^2)\,,\\
&= \frac{1}{\sqrt{-q_i}}\cosh^{-1}\frac{q_{ji}-2q_iu_{ji}^2}{\sqrt{q_{ji}^2-4q_iq_j}} && (q_i<0\,;\quad q_{ji}^2>4q_iq_j)\,,\\
&= \frac{1}{\sqrt{q_i}}\cos^{-1}\frac{q_{ji}-2q_iu_{ji}^2}{\sqrt{q_{ji}^2-4q_iq_j}} && (q_i>0\,;\quad q_{ji}^2>4q_iq_j)\,.
\end{aligned}\right\}\quad(1373)$$

From the foregoing equations it follows that *either of the two quantities ϕ_i or ϕ_j can be determined in terms of the other through the solution of a certain transcendental equation.*

The theorem we have just proved has an immediate application to the theory of the differential equation

$$\psi\frac{d\alpha}{dt}+2\alpha\frac{d\psi}{dt}+\frac{1}{2}\frac{d^3\psi}{dt^3}=0\,, \qquad (1374)$$

where α and ψ are functions of t. Equation (1374) can be regarded either as a first-order differential equation for α or as a third-order differential equation for ψ; it is from this latter point of view that we shall now consider this equation. From equation (1374) we derive the integral (cf. Eqs. [1037]-[1043])

$$\alpha=\frac{q}{\psi^2}-\frac{1}{\sqrt{\psi}}\frac{d^2}{dt^2}\sqrt{\psi}\,, \qquad (1375)$$

where q is an arbitrary constant. If we let $\psi = \phi^2$, we obtain

$$a = \frac{q}{\phi^4} - \frac{\ddot{\phi}}{\phi} . \tag{1376}$$

Consequently, *if any particular integral* $\psi(t) = \phi^2$ *of equation (1375) is known for some value of* q, *then the general solution of equation (1374) can be obtained in terms of the particular integral through a certain transcendental relation of the form (1372) or (1373).*

We shall finally consider some special cases:

(i) $q_i = 0$.—According to (1371), we now have

$$\int^t \frac{dt}{\phi_i^2} = \mp \int \frac{u_{ji} du_{ji}}{\sqrt{-q_j + q_{ji} u_{ji}^2}} , \tag{1377}$$

or, more explicitly,

$$\int^t \frac{dt}{\phi_i^2} = \mp \frac{1}{q_{ji}} \sqrt{-q_j + q_{ji} u_{ji}^2} . \tag{1378}$$

On the other hand, according to (1369), we have

$$\int^t \frac{dt}{\phi_j^2} = \pm \int \frac{du_{ij}}{\sqrt{q_{ij} - q_j u_{ij}^2}} ; \tag{1379}$$

or, after evaluating the integral on the right-hand side, we have

$$\left.\begin{aligned} \int^t \frac{dt}{\phi_j^2} &= \pm \frac{1}{\sqrt{q_j}} \sin^{-1} \sqrt{\frac{q_j}{q_{ij}}}\, u_{ij} && (q_j > 0 ; \quad q_{ij} > 0) , \\ &= \pm \frac{1}{\sqrt{-q_j}} \sinh^{-1} \sqrt{\frac{-q_j}{q_{ij}}}\, u_{ij} && (q_j < 0 ; \quad q_{ij} > 0) , \\ &= \pm \frac{1}{\sqrt{-q_j}} \cosh^{-1} \sqrt{\frac{q_j}{q_{ij}}}\, u_{ij} && (q_j < 0 ; \quad q_{ij} < 0) . \end{aligned}\right\} \tag{1380}$$

(ii) $q_i = q_j = 0$.—We then have

$$\int^t \frac{dt}{\phi_j^2} = \pm \frac{1}{\sqrt{q_{ij}}} u_{ij} ; \qquad \int^t \frac{dt}{\phi_i^2} = \mp \frac{1}{\sqrt{q_{ji}}} u_{ji} . \tag{1381}$$

(iii) $4q_i q_j = q_{ij}^2 = q_{ji}^2$.—As may be readily verified, we now have the relations

$$\left.\begin{aligned} 2\int^t \frac{dt}{\phi_j^2} &= \pm\frac{1}{\sqrt{-q_j}} \log\left(1 - \sqrt{\frac{q_j}{q_i}}\, u_{ij}^2\right), \\ 2\int^t \frac{dt}{\phi_i^2} &= \pm\frac{1}{\sqrt{-q_i}} \log\left(1 - \sqrt{\frac{q_i}{q_j}}\, u_{ji}^2\right). \end{aligned}\right\} \tag{1382}$$

64. *Spherical systems.*—We shall begin our discussion of the fourteen systems of equations obtained in § 62 by considering first the case of spherical symmetry, i.e., when

$$\alpha_1 = \alpha_2 = \alpha_3 = \alpha(t) \quad (\text{say}) . \tag{1383}$$

The systems I and II are identically satisfied. Hence,

$$f_{21},\ g_{21},\ h_{21},\ f_{40},\ g_{40},\ h_{40} \neq 0 \tag{1384}$$

and are arbitrary constants.

Consider next the system III. Of the three equations III (ii), only two are independent, and these can be written as (using [i])

$$\left.\begin{aligned} \frac{d^2}{dt^2}(f_{11} - g_{11}) + \alpha(f_{11} - g_{11}) &= 0 , \\ \frac{d^2}{dt^2}(2g_{11} + f_{11}) + \alpha(2g_{11} + f_{11}) &= 0 . \end{aligned}\right\} \tag{1385}$$

From the foregoing equations it readily follows that

$$\frac{d^2 f_{11}}{dt^2} + \alpha f_{11} = 0 ; \qquad \frac{d^2 g_{11}}{dt^2} + \alpha g_{11} = 0 . \tag{1386}$$

On the other hand, according to the first of the equations III (iii), we have

$$\frac{d^3 f_{11}}{dt^3} + 3\alpha \frac{df_{11}}{dt} + 2\alpha \frac{d}{dt}(g_{11} + h_{11}) - (h_{11} + g_{11}) \frac{d\alpha}{dt} = 0 ; \tag{1387}$$

or, using (i), we have

$$\frac{d^3 f_{11}}{dt^3} + \alpha \frac{df_{11}}{dt} + f_{11} \frac{d\alpha}{dt} = 0 , \tag{1388}$$

or, alternatively,

$$\frac{d}{dt}\left(\frac{d^2 f_{11}}{dt^2} + \alpha f_{11}\right) = 0 ; \tag{1389}$$

but this equation is satisfied identically in virtue of equation (1386). Similarly, the other two equations of III (iii) are also satisfied in virtue of equation (1386). Hence, f_{11}, g_{11}, h_{11} all satisfy the differential equation

$$\frac{d^2\phi}{dt^2} + \alpha\phi = 0 \quad (\phi = f_{11}, g_{11}, h_{11}) \tag{1390}$$

and are further restricted to satisfy the condition III (i).

When all the α's are equal, equations (ii) and (iii) of all of the systems IV–IX are identically the same and are satisfied in virtue of the equations (i) of these systems. Consequently, f_{20}, g_{20}, h_{20}, f_{30}, g_{30}, and h_{30} all satisfy the equation (1390).

Consider now the system X. According to equations (i) and (ii), we have

$$\beta_1 = \gamma_1 + \text{constant} \tag{1391}$$

and

$$\beta_1 = \frac{1}{2}\frac{df_{10}}{dt} + \beta_{10} ; \qquad \gamma_1 = \frac{1}{2}\frac{df_{10}}{dt} - \beta_{10} , \tag{1392}$$

where β_{10} is a constant. Equations (iii) and (iv) now take the forms

$$\frac{d^2\beta_1}{dt^2} + f_{10}\frac{d\alpha}{dt} + 2\alpha(\beta_1 + \gamma_1) = 0 \tag{1393}$$

and

$$\frac{d^2\gamma_1}{dt^2} + f_{10}\frac{d\alpha}{dt} + 2\alpha(\beta_1 + \gamma_1) = 0 . \tag{1394}$$

Subtracting equation (1394) from (1393), we obtain

$$\frac{d^2}{dt^2}(\beta_1 - \gamma_1) = 0\,, \tag{1395}$$

but this equation is satisfied in virtue of equation (1391). Adding next the two equations (1393) and (1394), we have

$$\frac{d^2}{dt^2}(\beta_1 + \gamma_1) + 2f_{10}\frac{d\alpha}{dt} + 4\alpha(\beta_1 + \gamma_1) = 0\,, \tag{1396}$$

or, using equation **X** (i), we have

$$f_{10}\frac{d\alpha}{dt} + 2\alpha\frac{df_{10}}{dt} + \frac{1}{2}\frac{d^3 f_{10}}{dt^3} = 0\,. \tag{1397}$$

Consequently,

$$\alpha = \frac{q}{f_{10}^2} - \frac{1}{\sqrt{f_{10}}}\frac{d^2}{dt^2}\sqrt{f_{10}}\,, \tag{1398}$$

where q is an arbitrary constant. We obtain similar results from the systems XI and XII. Finally, we have the systems XIII and XIV.

Summarizing the results obtained, we have: (i) a_0, b_0, c_0, f_{10}, g_{10}, and h_{10} all satisfy the differential equation

$$\psi\frac{d\alpha}{dt} + 2\alpha\frac{d\psi}{dt} + \frac{1}{2}\frac{d^3\psi}{dt^3} = 0 \qquad (\psi = a_0, b_0, c_0, f_{10}, g_{10}, h_{10})\,, \tag{1399}$$

or

$$\alpha = \frac{q}{\psi^2} - \frac{1}{\sqrt{\psi}}\frac{d^2}{dt^2}\sqrt{\psi}\,, \tag{1400}$$

where q is an arbitrary constant;[55] (ii) f_{11}, g_{11}, h_{11}, f_{20}, g_{20}, h_{20}, f_{30}, g_{30}, h_{30}, δ_1, δ_2, and δ_3 all satisfy the differential equation

$$\alpha = -\frac{\ddot{\phi}}{\phi} \qquad (\phi = f_{11}, g_{11}, h_{11}, f_{20}, g_{20}, h_{20}, f_{30}, g_{30}, h_{30}, \delta_1, \delta_2, \delta_3)\;; \tag{1401}$$

[55] The constant q will, in general, have different values when ψ stands successively for the quantities a_0, b_0, c_0, f_{10}, g_{10}, and h_{10}.

(iii)

$$\left.\begin{aligned}
\beta_1 &= \frac{1}{2}\frac{df_{10}}{dt} + \beta_{10}\,; & \gamma_1 &= \frac{1}{2}\frac{df_{10}}{dt} - \beta_{10}\,,\\
\beta_2 &= \frac{1}{2}\frac{dg_{10}}{dt} + \beta_{20}\,; & \gamma_2 &= \frac{1}{2}\frac{dg_{10}}{dt} - \beta_{20}\,,\\
\beta_3 &= \frac{1}{2}\frac{dh_{10}}{dt} + \beta_{30}\,; & \gamma_3 &= \frac{1}{2}\frac{dh_{10}}{dt} - \beta_{30}\,,
\end{aligned}\right\} \tag{1402}$$

where β_{10}, β_{20}, and β_{30} are constants; and, finally, (iv)

$$f_{21},\ g_{21},\ h_{21},\ f_{40},\ g_{40},\ h_{40} \neq 0 \tag{1403}$$

are further constants. We thus see that all the functions of time which are introduced in our general solutions for the coefficients of the velocity ellipsoid and for the Δ's are (under the conditions of our present problem) now determined in terms of the single function $\alpha(t)$. Further, according to the theorem proved in § 63, if any one of the eighteen quantities $(a_0, \ldots, \delta_3)$ is known as a function of time, then all the others can be determined in terms of this one function through certain transcendental relations of the forms (1372) or (1373).

If α is a constant, then the solutions for the various quantities can be explicitly written down. We have[56]

$$\left.\begin{aligned}
\psi_i &= A_i \sin 2\sqrt{\alpha}\,(t + t_i) + B_i & (i &= 1, \ldots, 6)\,,\\
\phi_i &= C_i \sin \sqrt{\alpha}\,(t + t_i') & (i &= 1, \ldots, 11)\,,\\
\beta_i &= \sqrt{\alpha}\,A_i \cos 2\sqrt{\alpha}\,(t + t_i) + \beta_{i0} & (i &= 1, \ldots, 3)\,,\\
\gamma_i &= \sqrt{\alpha}\,A_i \cos 2\sqrt{\alpha}\,(t + t_i) - \beta_{i0} & (i &= 1, \ldots, 3)\,,\\
f_{21},\ g_{21},\ h_{21},\ f_{40},\ g_{40},\ h_{40} &= \kappa_i & (i &= 1, \ldots, 6)\,,
\end{aligned}\right\} \tag{1404}$$

where A_i, B_i, C_i, t_i, t_i', β_{i0}, and κ_i are all constants. The general solution thus contains $6 \times 3 + 11 \times 2 + 3 + 6$, or forty-nine, constants of integration.[57]

[56] We have used $\psi_1, \ldots, \psi_6$ to represent f_{10}, g_{10}, h_{10}, a_0, b_0, and c_0, respectively; similarly, $\phi_1, \ldots, \phi_{11}$ represent f_{11}, g_{11}, f_{20}, $\ldots$, δ_3, respectively.

[57] This is true also when α is not a constant.

65. *Spheroidal systems.*—For spheroidal systems two of the α's are equal, and we shall assume that

$$\alpha_1 = \alpha_2 \neq \alpha_3 . \tag{1405}$$

We proceed now to a consideration of the fourteen systems of equations obtained in § 62.

From the systems I and II it readily follows that

$$f_{21} = g_{21} = h_{21} = f_{40} = g_{40} = 0 . \tag{1406}$$

Further,

$$h_{40} \neq 0 \tag{1407}$$

and is an arbitrary constant.

Consider next the system III quite generally, i.e., *without* using the fact that under our present conditions of the problem $\alpha_1 = \alpha_2$. Of the three equations (ii), only two are clearly independent. Choosing the first and the second of the equations (ii) as the independent ones, we can re-write them, using equation (i) in the forms

$$\left.\begin{aligned} &\frac{d^2}{dt^2}(f_{11} - g_{11}) + (\alpha_2 + \alpha_3 - \alpha_1)f_{11} - (\alpha_3 + \alpha_1 - \alpha_2)g_{11} = 0 , \\ &\frac{d^2}{dt^2}(g_{11} - h_{11}) + (\alpha_3 + \alpha_1 - \alpha_2)g_{11} - (\alpha_1 + \alpha_2 - \alpha_3)h_{11} = 0 . \end{aligned}\right\} \tag{1408}$$

On the other hand, from equations (iii) we readily obtain

$$\left.\begin{aligned} &\frac{d^3}{dt^3}(f_{11} - g_{11}) + (\alpha_2 + \alpha_3 - \alpha_1)\frac{df_{11}}{dt} - (\alpha_3 + \alpha_1 - \alpha_2)\frac{dg_{11}}{dt} \\ &\qquad + h_{11}\frac{d}{dt}(\alpha_1 - \alpha_2) + (f_{11} - g_{11})\frac{d\alpha_3}{dt} = 0 , \\ &\frac{d^3}{dt^3}(g_{11} - h_{11}) + (\alpha_3 + \alpha_1 - \alpha_2)\frac{dg_{11}}{dt} - (\alpha_1 + \alpha_2 - \alpha_3)\frac{dh_{11}}{dt} \\ &\qquad + f_{11}\frac{d}{dt}(\alpha_2 - \alpha_3) + (g_{11} - h_{11})\frac{d\alpha_1}{dt} = 0 . \end{aligned}\right\} \tag{1409}$$

Using (i), and after some further reductions, we find that the foregoing equations can be re-written as

$$\left.\begin{aligned} \frac{d}{dt}\left\{\frac{d^2}{dt^2}(f_{11}-g_{11})+(a_2+a_3-a_1)f_{11}-(a_3+a_1-a_2)g_{11}\right\}&=0\,,\\ \frac{d}{dt}\left\{\frac{d^2}{dt^2}(g_{11}-h_{11})+(a_3+a_1-a_2)g_{11}-(a_1+a_2-a_3)h_{11}\right\}&=0\,;\end{aligned}\right\}\quad(1410)$$

but these equations are identically satisfied in virtue of the equations (1408). Consequently, only one of the three equations (iii) is independent of the equations (ii). The single independent equation to which the equations (iii) are equivalent can be most conveniently expressed by adding the three equations (iii); we then find (using [i]) that

$$4a_1\frac{df_{11}}{dt}+4a_2\frac{dg_{11}}{dt}+4a_3\frac{dh_{11}}{dt}+f_{11}\frac{da_1}{dt}+g_{11}\frac{da_2}{dt}+h_{11}\frac{da_3}{dt}=0\,. \quad (1411)$$

Let us now put $a_1 = a_2$ in equations (1408) and (1411). We find (on some further reductions, using [i]) that

$$\frac{d^2}{dt^2}(f_{11}-g_{11})+a_3(f_{11}-g_{11})=0\,, \quad (1412)$$

$$\frac{d^2}{dt^2}(2g_{11}+f_{11})+2a_1g_{11}+(2a_1-a_3)f_{11}=0\,, \quad (1413)$$

and

$$4(a_3-a_1)\frac{dh_{11}}{dt}+h_{11}\frac{d}{dt}(a_3-a_1)=0\,. \quad (1414)$$

Multiplying equation (1413) by 2 and adding equation (1412) to the result, we find that

$$3\frac{d^2h_{11}}{dt^2}+(4a_1-a_3)h_{11}=0\,. \quad (1415)$$

Now, equation (1414) can be integrated to give

$$h_{11}^4(a_3-a_1)=\text{constant}\,, \quad (1416)$$

or

$$h_{11} = \frac{\text{constant}}{|\alpha_3 - \alpha_1|^{1/4}} . \tag{1417}$$

Equation (1417) enables the elimination of h_{11} from (1415). As the eliminant, we have the differential equation

$$3 \frac{d^2}{dt^2} \frac{1}{|\alpha_3 - \alpha_1|^{1/4}} + \frac{4\alpha_1 - \alpha_3}{|\alpha_3 - \alpha_1|^{1/4}} = 0 . \tag{1418}$$

The foregoing is an important differential equation, since it provides a relation between α_1 and α_3—quantities which were originally introduced as two independent functions of time. We shall return to this equation later, but meantime we shall continue with the discussion of the systems IV–XIV.

For $\alpha_1 = \alpha_2$, the systems IV and VIII provide equations identical in forms for f_{20} and g_{30}. It would therefore be sufficient to consider one of the two systems. Let us consider the system IV. Subtracting equation (iii) from twice equation (ii), we find that

$$3 \frac{d^3 f_{20}}{dt^3} + (4\alpha_1 - \alpha_3) \frac{df_{20}}{dt} + f_{20} \frac{d}{dt} (4\alpha_1 - \alpha_3) = 0 , \tag{1419}$$

or

$$\frac{d}{dt} \left\{ 3 \frac{d^2 f_{20}}{dt^2} + (4\alpha_1 - \alpha_3) f_{20} \right\} = 0 ; \tag{1420}$$

but this equation is identically satisfied in virtue of equation (i). On the other hand, adding the equations (ii) and (iii), we have

$$4(\alpha_3 - \alpha_1) \frac{df_{20}}{dt} + f_{20} \frac{d}{dt} (\alpha_3 - \alpha_1) = 0 . \tag{1421}$$

Equation (1421) is readily integrated, and we have

$$f_{20} = \frac{\text{constant}}{|\alpha_3 - \alpha_1|^{1/4}} . \tag{1422}$$

Similarly, from the system VIII we find that

$$g_{30} = \frac{\text{constant}}{|\alpha_3 - \alpha_1|^{1/4}} . \tag{1423}$$

Substituting (1422) (or [1423]) in equation (i) of system IV (or VIII), we again find that α_1 and α_3 are related according to the differential equation (1418).

Consider next the systems V and VII. These systems provide equations of identical forms for g_{20} and f_{30}. Further, these equations can be treated exactly as those for f_{20} and g_{30}. We thus find that

$$g_{20} = \frac{\text{constant}}{|\alpha_3 - \alpha_1|^{1/4}} ; \qquad f_{30} = \frac{\text{constant}}{|\alpha_3 - \alpha_1|^{1/4}} . \tag{1424}$$

The elimination of g_{20} and f_{30} from the equations (i), V and (i), VII leads, however, to the differential equation

$$3 \frac{d^2}{dt^2} \frac{1}{|\alpha_3 - \alpha_1|^{1/4}} + \frac{4\alpha_3 - \alpha_1}{|\alpha_3 - \alpha_1|^{1/4}} = 0 , \tag{1425}$$

which is different from, and inconsistent with, equation (1418). Consequently, *either* f_{20}, g_{30}, and h_{11} are nonzero, g_{20} and f_{30} vanish identically, and equation (1418) is valid *or* g_{20} and f_{30} are nonzero, f_{20}, g_{30}, and h_{11} vanish identically, and equation (1425) is valid.[58]

Turning next to the systems VI and IX, we see that the equations (ii) and (iii) of these systems are the same (since $\alpha_1 = \alpha_2$) and are further satisfied identically in virtue of the equations (i) of these systems. Thus, h_{20} and h_{30} both satisfy the differential equation

$$\frac{d^2\phi}{dt^2} + \alpha_1\phi = 0 \qquad (\phi = h_{20}, h_{30}) . \tag{1426}$$

Consider now the system X. Subtracting equation (iv) from (iii), we obtain

$$\frac{d^2}{dt^2} (\beta_1 - \gamma_1) + f_{10} \frac{d}{dt} (\alpha_3 - \alpha_1) + (\alpha_3 - \alpha_1)(\beta_1 + \gamma_1) = 0 ; \tag{1427}$$

[58] We notice here the correspondence between our present nonsteady-state treatment of the problem and the steady-state treatment of the same problem given in Part VIII (see particularly the discussion of the cases II$_b$ and II$_c$ in § 37).

or, using equation (i), we have

$$\frac{d^2}{dt^2}(\beta_1 - \gamma_1) + f_{10}\frac{d}{dt}(\alpha_3 - \alpha_1) + (\alpha_3 - \alpha_1)\frac{df_{10}}{dt} = 0\,, \quad (1428)$$

or, alternatively,

$$\frac{d}{dt}\left\{\frac{d}{dt}(\beta_1 - \gamma_1) + (\alpha_3 - \alpha_1)f_{10}\right\} = 0\,. \quad (1429)$$

But the foregoing equation is satisfied identically in virtue of equation (i). Again, adding equations (iii) and (iv), we have

$$\left.\begin{aligned}\frac{d^2}{dt^2}(\beta_1 + \gamma_1) + f_{10}\frac{d}{dt}(\alpha_1 + \alpha_3) + \alpha_3(\beta_1 + 3\gamma_1)&\\ + \alpha_1(\gamma_1 + 3\beta_1) = 0\,.&\end{aligned}\right\} \quad (1430)$$

Since

$$\left.\begin{aligned}\alpha_3(\beta_1 + 3\gamma_1) + \alpha_1(\gamma_1 + 3\beta_1) = 2(\alpha_1 + \alpha_3)(\beta_1 + \gamma_1)&\\ + (\alpha_1 - \alpha_3)(\beta_1 - \gamma_1)\,,&\end{aligned}\right\} \quad (1431)$$

we can re-write equation (1430) as

$$\left.\begin{aligned}\frac{d^3f_{10}}{dt^3} + f_{10}\frac{d}{dt}(\alpha_1 + \alpha_3) + 2(\alpha_1 + \alpha_3)\frac{df_{10}}{dt}&\\ + (\alpha_1 - \alpha_3)(\beta_1 - \gamma_1) = 0\,.&\end{aligned}\right\} \quad (1432)$$

Multiplying the foregoing equation by f_{10} and using (ii), we have

$$\left.\begin{aligned}\frac{d}{dt}\left\{f_{10}\frac{d^2f_{10}}{dt^2} - \frac{1}{2}\left(\frac{df_{10}}{dt}\right)^2\right\} + \frac{d}{dt}\{f_{10}^2(\alpha_1 + \alpha_3)\}&\\ + \frac{1}{2}\frac{d}{dt}(\beta_1 - \gamma_1)^2 = 0\,.&\end{aligned}\right\} \quad (1433)$$

Integrating equation (1433), we have

$$f_{10}\frac{d^2f_{10}}{dt^2} - \frac{1}{2}\left(\frac{df_{10}}{dt}\right)^2 + f_{10}^2(\alpha_1 + \alpha_3) + \tfrac{1}{2}(\beta_1 - \gamma_1)^2 = \text{constant}\,. \quad (1434)$$

Using (i), we can re-write (1434) more conveniently in the form

$$f_{10}\frac{d^2f_{10}}{dt^2}+f_{10}^2(\alpha_1+\alpha_3) = 2\beta_1\gamma_1+\text{constant}\,. \tag{1435}$$

Equation (1435), together with the equations (i) and (ii), are our three independent equations for f_{10}, β_1, and γ_1. In the same way, the system XI provides us with a similar set of equations for g_{10}, β_2, and γ_2.

Consider next the system XII. Since $\alpha_1 = \alpha_2$, this system is formally the same as when we have spherical symmetry. Consequently, we have (cf. Eq. [1392])

$$\beta_3 = \frac{1}{2}\frac{dh_{10}}{dt}+\beta_{30}\,; \qquad \gamma_3 = \frac{1}{2}\frac{dh_{10}}{dt}-\beta_{30}\,, \tag{1436}$$

where β_{30} is a constant. Further, the differential equation for h_{10} is

$$2\alpha_1\frac{dh_{10}}{dt}+h_{10}\frac{d\alpha_1}{dt}+\frac{1}{2}\frac{d^3h_{10}}{dt^3} = 0\,. \tag{1437}$$

And, finally, we have the systems XIII and XIV.

Collecting the results of our discussion so far, we have shown that (i)

$$f_{21} = g_{21} = h_{21} = f_{40} = g_{40} = 0\,; \qquad h_{40} \neq 0\,; \tag{1438}$$

(ii) h_{20}, h_{30}, δ_1, and δ_2 all satisfy the differential equation

$$\alpha_1 = -\frac{\ddot{\phi}}{\phi} \qquad (\phi = h_{20},\, h_{30},\, \delta_1,\, \delta_2)\,; \tag{1439}$$

(iii) $(f_{11}-g_{11})$ and δ_3 satisfy the differential equation

$$\alpha_3 = -\frac{\ddot{\varphi}}{\varphi} \qquad \{\varphi = (f_{11}-g_{11}),\, \delta_3\}\,; \tag{1440}$$

(iv) *either*

$$f_{20} = \frac{\text{constant}}{|\alpha_3-\alpha_1|^{1/4}}\,; \qquad g_{30} = \frac{\text{constant}}{|\alpha_3-\alpha_1|^{1/4}}\,; \qquad h_{11} = \frac{\text{constant}}{|\alpha_3-\alpha_1|^{1/4}}\,, \tag{1441}$$

$$f_{30} = g_{20} \equiv 0\,, \tag{1442}$$

and

$$3\frac{d^2}{dt^2}\frac{1}{|\alpha_3-\alpha_1|^{1/4}}+\frac{4\alpha_1-\alpha_3}{|\alpha_3-\alpha_1|^{1/4}}=0\,, \tag{1443}$$

or

$$g_{20}=\frac{\text{constant}}{|\alpha_3-\alpha_1|^{1/4}}\,;\qquad f_{30}=\frac{\text{constant}}{|\alpha_3-\alpha_1|^{1/4}}\,, \tag{1444}$$

$$f_{20}=g_{30}=h_{11}\equiv 0\,, \tag{1445}$$

and

$$3\frac{d^2}{dt^2}\frac{1}{|\alpha_3-\alpha_1|^{1/4}}+\frac{4\alpha_3-\alpha_1}{|\alpha_3-\alpha_1|^{1/4}}=0\,; \tag{1446}$$

(v) a_0, b_0, and h_{10} all satisfy the differential equation

$$\psi\frac{d\alpha_1}{dt}+2\alpha_1\frac{d\psi}{dt}+\frac{1}{2}\frac{d^3\psi}{dt^3}=0\qquad(\psi=a_0,\,b_0,\,h_{10})\,, \tag{1447}$$

or

$$\alpha_1=\frac{q_1}{\psi^2}-\frac{1}{\sqrt{\psi}}\frac{d^2}{dt^2}\sqrt{\psi}\,, \tag{1448}$$

where q_1 is an arbitrary constant (which will be different when ψ stands successively for a_0, b_0, and h_{10}); (vi) c_0 satisfies the differential equation

$$c_0\frac{d\alpha_3}{dt}+2\alpha_3\frac{dc_0}{dt}+\frac{1}{2}\frac{d^3c_0}{dt^3}=0\,, \tag{1449}$$

or

$$\alpha_3=\frac{q_2}{c_0^2}-\frac{1}{\sqrt{c_0}}\frac{d^2}{dt^2}\sqrt{c_0}\,, \tag{1450}$$

where q_2 is an arbitrary constant; (vii)

$$\beta_3=\frac{1}{2}\frac{dh_{10}}{dt}+\beta_{30}\,;\qquad \gamma_3=\frac{1}{2}\frac{dh_{10}}{dt}-\beta_{30}\,, \tag{1451}$$

here β_{30} is a constant; (viii)

$$\frac{d}{dt}(\beta_1 - \gamma_1) = (\alpha_1 - \alpha_3)f_{10}\,; \qquad \frac{df_{10}}{dt} = \beta_1 + \gamma_1\,, \tag{1452}$$

$$f_{10}\frac{d^2 f_{10}}{dt^2} + f_{10}^2(\alpha_1 + \alpha_3) = 2\beta_1\gamma_1 + \text{constant}\,; \tag{1453}$$

and, finally, (ix)

$$\frac{d}{dt}(\beta_2 - \gamma_2) = (\alpha_3 - \alpha_1)g_{10}\,; \qquad \frac{dg_{10}}{dt} = \beta_2 + \gamma_2\,, \tag{1454}$$

$$g_{10}\frac{d^2 g_{10}}{dt^2} + g_{10}^2(\alpha_1 + \alpha_3) = 2\beta_2\gamma_2 + \text{constant}\,. \tag{1455}$$

We shall now show how the various quantities can all be expressed in terms of one single arbitrary function of time. Consider first the case when f_{20}, g_{30}, and h_{11} do not all vanish identically. We then have as our fundamental differential equation (cf. Eq. [1443]):

$$3|\alpha_3 - \alpha_1|^{1/4}\frac{d^2}{dt^2}\frac{1}{|\alpha_3 - \alpha_1|^{1/4}} + 4\alpha_1 - \alpha_3 = 0\,. \tag{1456}$$

Now, equation (1456) can be re-written in either of the forms

$$\left.\begin{aligned} \alpha_3 &= \tfrac{4}{3}(\alpha_3 - \alpha_1) - |\alpha_3 - \alpha_1|^{1/4}\frac{d^2}{dt^2}\frac{1}{|\alpha_3 - \alpha_1|^{1/4}}\,, \\ \alpha_1 &= \tfrac{1}{3}(\alpha_3 - \alpha_1) - |\alpha_3 - \alpha_1|^{1/4}\frac{d^2}{dt^2}\frac{1}{|\alpha_3 - \alpha_1|^{1/4}}\,. \end{aligned}\right\} \tag{1457}$$

Let us now introduce the variable Φ defined by

$$\Phi^4 = \frac{3}{|\alpha_3 - \alpha_1|}\,. \tag{1458}$$

From equation (1457) we now obtain the relations

$$\alpha_1 = \pm\frac{1}{\Phi^4} - \frac{\ddot{\Phi}}{\Phi}\,; \qquad \alpha_3 = \pm\frac{4}{\Phi^4} - \frac{\ddot{\Phi}}{\Phi}\,. \tag{1459}$$

where the plus or the minus sign should be taken according as α_3 is greater or less than α_1. Comparing (1459) with the equations (1439), (1440), (1448), and (1450), we see that we can immediately apply the theorems of § 63 to express the various quantities a_0, b_0, etc., all in terms of the single function $\Phi(t)$, which is left unspecified. Thus, according to equations (1439) and (1459), we have

$$-\frac{\ddot{\phi}}{\phi} = \pm\frac{1}{\Phi^4} - \frac{\ddot{\Phi}}{\Phi} . \tag{1460}$$

Hence, using the results of § 62, we have (cf. Eqs. [1380])

$$\left.\begin{aligned} \int^t \frac{dt}{\Phi^2} &= \pm \sin^{-1}\frac{\phi}{q\Phi} && (\alpha_3 > \alpha_1) , \\ &= \pm \sinh^{-1}\frac{\phi}{q\Phi} \quad \text{or} \quad \pm \cosh^{-1}\frac{\phi}{q\Phi} && (\alpha_3 < \alpha_1) , \end{aligned}\right\} \tag{1461}$$

where q is some arbitrary constant. Equation (1461) can be written alternatively in the form

$$\phi = q\Phi \left\{\begin{matrix} \sin \\ \sinh \\ \cosh \end{matrix}\right\} \int^t \frac{dt}{\Phi^2} \quad \left\{\begin{matrix} \alpha_3 > \alpha_1 \\ \alpha_3 < \alpha_1 \\ \alpha_3 < \alpha_1 \end{matrix}\right\} ; \tag{1462}$$

or, using our definition for Φ, we have

$$\left.\begin{aligned} \phi = \frac{\text{constant}}{|\alpha_3 - \alpha_1|^{1/4}} \left\{\begin{matrix} \sin \\ \sinh \\ \cosh \end{matrix}\right\} \frac{1}{\sqrt{3}}\int^t \sqrt{|\alpha_3 - \alpha_1|}\, dt \\ (\phi = h_{20}, h_{30}, \delta_1, \delta_2) . \end{aligned}\right\} \tag{1463}$$

Similarly, from equations (1440) and (1459) we obtain

$$\left.\begin{aligned} \varphi = \frac{\text{constant}}{|\alpha_3 - \alpha_1|^{1/4}} \left\{\begin{matrix} \sin \\ \sinh \\ \cosh \end{matrix}\right\} \frac{2}{\sqrt{3}}\int^t \sqrt{|\alpha_3 - \alpha_1|}\, dt \\ (\varphi = f_{11} - g_{11}, \delta_3) . \end{aligned}\right\} \tag{1464}$$

Again, combining the equations (1448), (1450), and (1459) appropriately, we can express a_0, b_0, c_0, and h_{10} in terms of Φ by relations of the forms (1372). Finally, it is clear that equations (1452)–(1455) will enable us to determine β_1, γ_1, β_2, γ_2, f_{10}, and g_{10} in terms of Φ.

The case when f_{30} and g_{20} do not vanish identically can be treated similarly. Under these circumstances our fundamental differential equation is (1446). From this equation we readily obtain

$$\alpha_1 = \mp \frac{4}{\Phi^4} - \frac{\ddot{\Phi}}{\Phi}\,; \qquad \alpha_3 = \mp \frac{1}{\Phi^4} - \frac{\ddot{\Phi}}{\Phi}\,, \tag{1465}$$

where Φ is defined as before; further, in (1465) we have the plus or the minus sign according as α_1 is greater or less than α_3.[59] Again, by combining the equations (1439), (1440), (1448), (1450), and (1465) appropriately, we can express all the quantities in terms of the single function Φ.

This completes the proof of the statement made at the outset, namely, that the general solution involves only one arbitrary function of time. Apart from this single arbitrariness, the solution is determinate.

66. *Spheroidal systems characterized by constant values of α_1 and α_3.*—If α_1 and α_3 are constants, the solutions for the various quantities can be explicitly written down. According to the equations (1438)–(1451), we now have

$$\text{(i)} \qquad f_{21} = g_{21} = h_{21} = f_{40} = g_{40} = 0\,; \qquad h_{40} = \text{constant}\,; \tag{1466}$$

$$\text{(ii)} \qquad \phi_i = A_i \sin \sqrt{\alpha_1}\,(t - t_i) \qquad (i = 1, \ldots, 4)\,, \tag{1467}$$

where the ϕ_i's stand for the quantities h_{20}, h_{30}, δ_1, and δ_2;

$$\text{(iii)} \qquad \varphi_i = A_i' \sin \sqrt{\alpha_3}\,(t - t_i') \qquad (i = 1, 2)\,, \tag{1468}$$

where φ_1 and φ_2 stand for $(f_{11} - g_{11})$ and δ_3;

$$\text{(iv)} \qquad \left\{ \begin{array}{lll} f_{20} = g_{30} = h_{11} = g_{20} = f_{30} = 0 & & (\alpha_1 \neq \tfrac{1}{4}\alpha_3, \neq 4\alpha_3)\,, \\ f_{20}, g_{30}, h_{11} \neq 0\,; & g_{20} = f_{30} = 0 & (4\alpha_1 = \alpha_3)\,, \\ f_{20} = g_{30} = h_{11} = 0\,; & g_{20}, f_{30} \neq 0 & (\alpha_1 = 4\alpha_3)\,; \end{array} \right\} \tag{1469}$$

$$\text{(v)} \qquad \psi_i = B_i \sin 2\sqrt{\alpha_1}\,(t - t_i'') + C_i \qquad (i = 1, 2, 3)\,, \tag{1470}$$

[59] It will be noticed that this is contrary to the situation we encountered when considering Eq. (1443).

where ψ_1, ψ_2, and ψ_3 stand, respectively, for a_0, b_0, and h_{10};

$$\text{(vi)} \qquad c_0 = B_4 \sin 2\sqrt{\alpha_3}\,(t - t_4'') + C_4 \,; \tag{1471}$$

$$\text{(vii)} \qquad \left\{\begin{aligned} \beta_3 &= \sqrt{\alpha_1}\, B_3 \sin 2\sqrt{\alpha_1}\,(t - t_3'') + \beta_{30} \,, \\ \gamma_3 &= \sqrt{\alpha_1}\, B_3 \sin 2\sqrt{\alpha_1}\,(t - t_3'') - \beta_{30} \,. \end{aligned}\right\} \tag{1472}$$

In the foregoing equations the quantities A_i, A_i', B_i, C_i, t_i, t_i', t_i'', and β_{30} are all arbitrary constants. Finally, we have to consider the equations (1452)–(1455) for β_1, γ_1, f_{10}, β_2, γ_2, h_{10}. However, these equations are best considered in their original forms (systems X and XI, § 62). Consider the system X. Since α_1 and α_3 are assumed to be constants, these equations take the forms

$$\frac{d}{dt}(\beta_1 - \gamma_1) = (\alpha_1 - \alpha_3)f_{10} \,; \qquad \frac{df_{10}}{dt} = \beta_1 + \gamma_1 \,, \tag{1473}$$

$$\left.\begin{aligned} \frac{d^2\beta_1}{dt^2} + (\alpha_1 + \alpha_3)\beta_1 + 2\alpha_3\gamma_1 &= 0 \,, \\ \frac{d^2\gamma_1}{dt^2} + 2\alpha_1\beta_1 + (\alpha_1 + \alpha_3)\gamma_1 &= 0 \,. \end{aligned}\right\} \tag{1474}$$

The equations (1474) represent a pair of simultaneous linear differential equations with constant coefficients and can be solved by standard methods. We find

$$\left.\begin{aligned} \beta_1 &= \sqrt{\alpha_3}\,[D_1 \sin(\sqrt{\alpha_3} + \sqrt{\alpha_1})t + D_2 \sin(\sqrt{\alpha_3} - \sqrt{\alpha_1})t] \,, \\ \gamma_1 &= \sqrt{\alpha_1}\,[D_1 \sin(\sqrt{\alpha_3} + \sqrt{\alpha_1})t - D_2 \sin(\sqrt{\alpha_3} - \sqrt{\alpha_1})t] \,, \end{aligned}\right\} \tag{1475}$$ [60]

[60] The quantities $-(\sqrt{\alpha_3} + \sqrt{\alpha_1})^2$ and $-(\sqrt{\alpha_3} - \sqrt{\alpha_1})^2$ are the characteristic roots of the matrix

$$\begin{vmatrix} \alpha_1 + \alpha_3 & 2\alpha_3 \\ 2\alpha_1 & \alpha_1 + \alpha_3 \end{vmatrix} \tag{1475'}$$

which is associated with the linear system (1474).

where D_1 and D_2 are constants. From the foregoing solutions for β_1 and γ_1 we obtain

$$\left.\begin{aligned}\beta_1 + \gamma_1 &= D_1(\sqrt{\alpha_3} + \sqrt{\alpha_1}) \sin (\sqrt{\alpha_3} + \sqrt{\alpha_1})t \\ &\qquad + D_2(\sqrt{\alpha_3} - \sqrt{\alpha_1}) \sin (\sqrt{\alpha_3} - \sqrt{\alpha_1})t\,, \\ \beta_1 - \gamma_1 &= D_1(\sqrt{\alpha_3} - \sqrt{\alpha_1}) \sin (\sqrt{\alpha_3} + \sqrt{\alpha_1})t \\ &\qquad + D_2(\sqrt{\alpha_3} + \sqrt{\alpha_1}) \sin (\sqrt{\alpha_3} - \sqrt{\alpha_1})t\,.\end{aligned}\right\} \quad (1476)$$

From equations (1473) and (1476) we readily find that

$$f_{10} = -[D_1 \cos (\sqrt{\alpha_3} + \sqrt{\alpha_1})t + D_2 \cos (\sqrt{\alpha_3} - \sqrt{\alpha_1})t]\,. \quad (1477)$$

Similarly from the system XI (§ 62) we find

$$\left.\begin{aligned}\beta_2 &= \sqrt{\alpha_1}\,[D_1' \sin (\sqrt{\alpha_3} + \sqrt{\alpha_1})t - D_2' \sin (\sqrt{\alpha_3} - \sqrt{\alpha_1})t]\,, \\ \gamma_2 &= \sqrt{\alpha_3}\,[D_1' \sin (\sqrt{\alpha_3} + \sqrt{\alpha_1})t + D_2' \sin (\sqrt{\alpha_3} - \sqrt{\alpha_1})t]\,, \\ g_{10} &= -\quad [D_1' \cos (\sqrt{\alpha_3} + \sqrt{\alpha_1})t + D_2' \cos (\sqrt{\alpha_3} - \sqrt{\alpha_1})t]\,,\end{aligned}\right\} \quad (1478)$$

where D_1' and D_2' are two further constants of integration. Thus, when α_1 and α_3 are constants and $\alpha_3 \neq 4\alpha_1$ or $\frac{1}{4}\alpha_1$, the general solution involves thirty constants of integration. If $\alpha_3 = 4\alpha_1$ or $\alpha_1 = 4\alpha_3$, then, according to equation (1469), we have three, respectively two, additional constants of integration.

67. *The evolution of spheroidal systems.*—As is well known, the potential inside a homogeneous spheroidal distribution of mass is expressible in the form

$$\mathfrak{V} = \tfrac{1}{2}[\alpha_1(x^2 + y^2) + \alpha_3 z^2] + \mathfrak{V}_0(t)\,. \quad (1479)$$

The quantities α_1 and α_3 are related to the density, ρ, of the spheroid by

$$\alpha_1 = \nu\rho G\,; \qquad \alpha_3 = \mu\rho G\,, \quad (1480)$$

where G is the constant of gravitation and ν and μ are two positive numbers depending only on the ratio κ of the major to the minor axis of the homogeneous spheroid:[61]

$$\left.\begin{aligned} 2\nu + \mu &= 4\pi \,, \\ 2\nu\kappa^2 + \mu &= \frac{4\pi\kappa^2}{\sqrt{\kappa^2 - 1}} \tan^{-1} \sqrt{\kappa^2 - 1} \,. \end{aligned}\right\} \tag{1481}$$

For very flat systems $\kappa \gg 1$, and we can write

$$2\nu + \mu = 4\pi \,; \qquad 2\nu\kappa^2 + \mu \simeq 2\pi^2\kappa \,; \tag{1482}$$

or, in the limit $\kappa \to \infty$, we have

$$\nu = \pi^2\kappa^{-1} + O(\kappa^{-2}) \,; \qquad \mu = 4\pi + O(\kappa^{-1}) \,. \tag{1483}$$

Correspondingly, for α_1 and α_3 we have

$$\alpha_1 \simeq \pi^2\kappa^{-1}\rho G \,; \qquad \alpha_3 \simeq 4\pi\rho G \,. \tag{1484}$$

Consequently, for very flat systems

$$\frac{\alpha_3}{\alpha_1} \simeq \frac{4}{\pi}\kappa = 1.2732\kappa \,. \tag{1485}$$ [62]

Now, our discussion of the evolution of spheroidal systems hinges on the differential equation

$$3(\alpha_3 - \alpha_1)^{1/4} \frac{d^2}{dt^2} \frac{1}{(\alpha_3 - \alpha_1)^{1/4}} + 4\alpha_1 - \alpha_3 = 0 \qquad (\alpha_3 > \alpha_1) \,. \tag{1486}$$

We have thus only one relation between the two quantities α_1 and α_3 which, within the framework of our fundamental postulates, should be regarded as independent. Consequently, to draw unambiguous conclusions concerning the evolution of the system, we should supplement equation (1486) by some other "extraneous" relation between

[61] See, e.g., E. J. Routh, *Analytical Statics*, **2**, 106–116, Cambridge, England, 1922.

[62] It is thus seen that the ratio of the axes of the potential spheroid is proportional to the *square root* of the ratio of the axes of the density-spheroid.

a_1 and a_3; but within the framework of our present theory we cannot do this uniquely. However, in spite of this circumstance, equation (1486) still enables us to draw certain conclusions which have definite evolutionary significance.

i) *The stability of stellar systems,* E7.07, *along the line* $a_3 = 4a_1$ *in the* (a_1, a_3) *plane.*—We have already seen in Part VII, § 34, and in Part VIII, §37, case II$_b$, that under steady-state circumstances a critical case arises when $a_3 = 4a_1$. This is, of course, in agreement

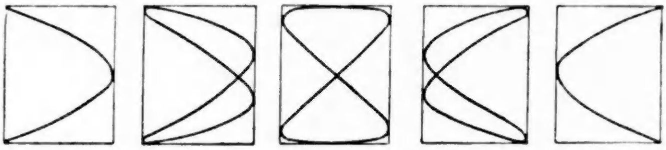

FIG. 5.—Lissajous' curves

with equation (1486), since, when a_1 and a_3 are constants, $a_3 = 4a_1$. Consequently, these critical stationary spheroids are represented by points along the line $4a_1 - a_3 = 0$ in the (a_1, a_3) plane (see Fig. 6). We shall now examine the stability with respect to oscillations of these stationary spheroids. To do this, we shall assume that

$$\left.\begin{aligned} a_1 &= a_0 + a_{11} \sin \nu_1(t - t_1) \,, \\ a_3 &= 4a_0 + a_{31} \sin \nu_3(t - t_3) \,, \end{aligned}\right\} \tag{1487}$$

where a_0, a_{11}, a_{31}, ν_1, ν_3, t_1, and t_3 are constants arbitrary in the first instance. It is further assumed that

$$\left|\frac{a_{11}}{a_0}\right| \ll 1 \; ; \qquad \left|\frac{a_{31}}{a_0}\right| \ll 1 \; ; \qquad a_0 > 0 \, . \tag{1488}$$

Solutions of the form (1487) correspond to stable oscillations about the equilibrium configuration $(a_0, 4a_0)$ if ν_1 and ν_2 are both real. On the other hand, if ν_1 and/or ν_2 are/is imaginary, then the configurations are unstable against small oscillations.

According to equations (1487), we have (neglecting quantities of the second order of smallness)

$$\left.\begin{aligned}(\alpha_3 - \alpha_1)^{-1/4} = (3\alpha_0)^{-1/4}\Bigg[1 - \frac{1}{12\alpha_0}\{\alpha_{31} \sin \nu_3(t - t_3) \\ - \alpha_{11} \sin \nu_1(t - t_1)\}\Bigg] .\end{aligned}\right\} \tag{1489}$$

Introducing equation (1489) in (1486), we have (again only to the first order of smallness)

$$\left.\begin{aligned}\frac{1}{4\alpha_0}\{\nu_3^2\alpha_{31} \sin \nu_3(t - t_3) - \nu_1^2\alpha_{11} \sin \nu_1(t - t_1)\} \\ + 4\alpha_{11} \sin \nu_1(t - t_1) - \alpha_{31} \sin \nu_3(t - t_3) = 0 .\end{aligned}\right\} \tag{1490}$$

We have now to distinguish between the two following cases:

Case (a), $\nu_1 \neq \nu_3$.—From equation (1490) we now infer that

$$\nu_1^2 = 16\alpha_0 ; \qquad \nu_3^2 = 4\alpha_0 . \tag{1491}$$

Thus, ν_1 and ν_3 are both real; further, α_{31} and α_{11} are arbitrary except for the condition that they are to be quantities of the first order of smallness, compared to α_0. Combining equations (1487) and (1491), we have

$$\left.\begin{aligned}\alpha_1 &= \alpha_0 + \alpha_{11} \sin 4\sqrt{\alpha_0}(t - t_1) , \\ \alpha_3 &= 4\alpha_0 + \alpha_{31} \sin 2\sqrt{\alpha_0}(t - t_3) .\end{aligned}\right\} \tag{1492}$$

We can re-write the foregoing as

$$\left.\begin{aligned}\alpha_1 - \alpha_0 &= \alpha_{11} \sin 4\sqrt{\alpha_0}(t - t_1) , \\ \alpha_3 - 4\alpha_0 &= \alpha_{31} \sin 2\sqrt{\alpha_0}(t - t_3) .\end{aligned}\right\} \tag{1493}$$

Thus, *in the* (α_1, α_3) *plane the motion of the representative point about the equilibrium state* $(\alpha_0, 4\alpha_0)$ *is expressible as the superposition of two simple harmonic vibrations in directions at right angles; further, the frequency of vibration along the* α_1*-axis is twice that along the* α_3*-axis.* Consequently, the curves defining the motion in the (α_1, α_3) plane

are Lissajous' curves (see Fig. 5).[63] It is also clear that during such oscillations the system does not, in general, pass through the equilibrium state $(\alpha_0, 4\alpha_0)$. In this respect the stellar system behaves more like a spherical, than like a simple, pendulum.

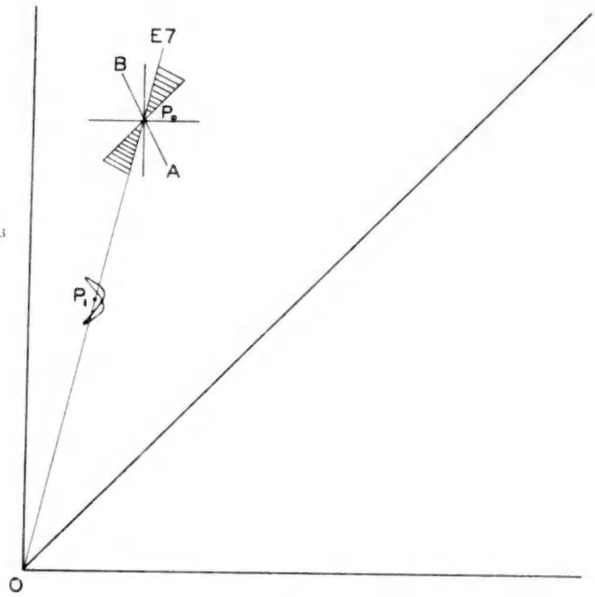

FIG. 6.—The stability of spheroidal systems represented on the line $\alpha_3 = 4\alpha_1$. If the straight line representing the oscillations of α_1 and α_3 (with the same frequency and phase) has a slope falling inside the shaded region, the oscillations are unstable; otherwise such oscillations represent a stable mode of vibration. The other possible mode of vibration (represented by the Lissajous' curve about P_1) is always stable.

Case (b), $\nu_1 = \nu_2 = \nu_0$. From equation (1490) we now conclude that

$$\left.\begin{aligned} \nu_0^2(\alpha_{31}\cos\nu_0 t_3 - \alpha_{11}\cos\nu_0 t_1) &= 4\alpha_0(\alpha_{31}\cos\nu_0 t_3 - 4\alpha_{11}\cos\nu_0 t_1)\,, \\ \nu_0^2(\alpha_{31}\sin\nu_0 t_3 - \alpha_{11}\sin\nu_0 t_1) &= 4\alpha_0(\alpha_{31}\sin\nu_0 t_3 - 4\alpha_{11}\sin\nu_0 t_1)\,. \end{aligned}\right\} \quad (1494)$$

[63] See, e.g., H. Lamb, *The Dynamical Theory of Sound*, §18, pp. 48–52, Cambridge, England, 1931.

The foregoing equations can be re-written alternatively in the form

$$\left.\begin{aligned} \nu_0^2 &= \frac{4\alpha_0(\alpha_{31} \cos \nu_0 t_3 - 4\alpha_{11} \cos \nu_0 t_1)}{(\alpha_{31} \cos \nu_0 t_3 - \alpha_{11} \cos \nu_0 t_1)} , \\ &= \frac{4\alpha_0(\alpha_{31} \sin \nu_0 t_3 - 4\alpha_{11} \sin \nu_0 t_1)}{(\alpha_{31} \sin \nu_0 t_3 - \alpha_{11} \sin \nu_0 t_1)} . \end{aligned}\right\} \tag{1495}$$

From the foregoing equations it readily follows that

$$\frac{\cos \nu_0 t_1}{\cos \nu_0 t_3} = \frac{\sin \nu_0 t_1}{\sin \nu_0 t_3} , \tag{1496}$$

or, alternatively,

$$\sin \nu_0(t_3 - t_1) = 0 . \tag{1497}$$

Equation (1497) implies that *the oscillations of* α_3 *and* α_1 *are in phase.* There is therefore no loss of generality if we set $t_1 = t_3 = t_0$. We can then write

$$\left.\begin{aligned} \alpha_1 &= \alpha_0 + \alpha_{11} \sin \nu_0(t - t_0) , \\ \alpha_3 &= 4\alpha_0 + \alpha_{31} \sin \nu_0(t - t_0) , \end{aligned}\right\} \tag{1498}$$

where, according to equations (1495)–(1497),

$$\nu_0^2 = \frac{4\alpha_0(\alpha_{31} - 4\alpha_{11})}{\alpha_{31} - \alpha_{11}} . \tag{1499}$$

According to equations (1498), we have

$$\frac{\alpha_3 - 4\alpha_0}{\alpha_1 - \alpha_0} = \frac{\alpha_{31}}{\alpha_{11}} , \tag{1500}$$

which implies that *the representative point in the* (α_1, α_3) *plane moves along a straight line through the equilibrium state* $(\alpha_0, 4\alpha_0)$ *with a slope* α_{31}/α_{11}. On the other hand, according to (1499),

$$\left.\begin{aligned} &\nu_0^2 < 0 \quad \text{if} \quad \frac{\alpha_{31}}{\alpha_{11}} < 4 \quad \text{and} \quad \frac{\alpha_{31}}{\alpha_{11}} > 1 ; \\ &\text{otherwise} \\ &\nu_0^2 > 0 . \end{aligned}\right\} \tag{1501}$$

Consequently, *the solution (1498) corresponds to unstable oscillations if the straight line (1500) defining the motion of the representative point has a slope greater than* 1 *and less than* 4. This situation is made explicit in Figure 6, where the directions of instability are indicated.

We can now summarize the results of our discussion so far, as follows:

For small oscillations about $(\alpha_0, 4\alpha_0)$, α_1 and α_3 oscillate with frequencies which are in the ratio 2 : 1 or 1 : 1; in the former case, the frequencies are real and the motion of the representative point in the (α_1, α_3) plane is a Lissajous' curve; in the latter case, α_1 and α_3 must oscillate in phase, and the frequency of oscillation becomes imaginary if, and only if, the straight line through the equilibrium state $(\alpha_0, 4\alpha_0)$ defining the motion of the representative point has a slope greater than 1 and less than 4. We thus see that stellar systems, the equilibrium states of which are represented by points on the line $\alpha_3 = 4\alpha_1$, are characterized by both stable and unstable modes of vibration.

Finally, we may notice that, since the systems represented on the line $\alpha_3 = 4\alpha_1$ are further characterized by a value of $\kappa = 3.410$ (cf. Eq. [827]), we may denote them by the symbol E7.07.[64]

ii) *The evolution of very flat spheroidal systems.*—Let us first consider the case of "infinitely flat" spheroidal systems. Then, according to equation (1485),

$$\frac{\alpha_3}{\alpha_1} \to \infty \,, \tag{1502}$$

and we can neglect α_1, compared to α_3. Under these circumstances our fundamental differential equation (1486) reduces to

$$3\frac{d^2Q_0}{dt^2} = \frac{1}{Q_0^3} \,, \tag{1503}$$

where

$$Q_0^4 = \alpha_3^{-1} \,. \tag{1504}$$

[64] The number attached to the symbol "E" is ten times the ellipticity (cf. E. Hubble, *The Realm of the Nebulae*, p. 41, New Haven, Yale University Press, 1936). In our case, therefore, the number to be attached to "E" is 10 × (3.410 − 1.000/3.410) = 7.07.

A first integral of equation (1503) is readily obtained by multiplying this equation by dQ_0/dt and integrating. We find in this manner that

$$3\left(\frac{dQ_0}{dt}\right)^2 = C - \frac{1}{Q_0^2}, \tag{1505}$$

where C is a constant of integration. The foregoing equation can be expressed alternatively in the form

$$\frac{Q_0 dQ_0}{\sqrt{CQ_0^2 - 1}} = \pm \frac{1}{\sqrt{3}}\, dt, \tag{1506}$$

in which form the equation admits of immediate integration. We have

$$\sqrt{CQ_0^2 - 1} = \pm \frac{C}{\sqrt{3}}(t - t_0), \tag{1507}$$

where t_0 is a second constant of integration. From (1507) we derive

$$Q_0^2 = \frac{1}{C}\left[1 + \frac{C^2}{3}(t - t_0)^2\right]; \tag{1508}$$

or, returning to the variable α_3 (cf. Eq. [1504]), we have

$$\alpha_3 = \frac{C^2}{\left[1 + \frac{C^2}{3}(t - t_0)^2\right]^2}. \tag{1509}$$

According to (1509),

$$\alpha_3 = \alpha_3(0) = C^2; \qquad t = t_0. \tag{1510}$$

Equation (1509) now takes the form

$$\alpha_3 = \frac{\alpha_3(0)}{\left[1 + \frac{\alpha_3(0)}{3}(t - t_0)^2\right]^2}. \tag{1511}$$

Let us now introduce the new variable τ defined by

$$\tau = \frac{\sqrt{\alpha_3(0)}}{3}(t - t_0) ; \tag{1512}$$

in other words, τ measures the time from the epoch $t = t_0$ in units of $\sqrt{3/\alpha_3(0)}$. From equation (1484) it follows that the explicit expression for the unit of time adopted is

$$\sqrt{\frac{3}{\alpha_3(0)}} = \sqrt{\frac{3}{4\pi\rho_0 G}} , \tag{1513}$$

where ρ_0 is the density at $t = t_0$. Numerically, the unit of time (1513) is found to be

$$\frac{7.27 \times 10^6}{\sqrt{\rho_0^*}} \text{ years} , \tag{1514}$$

where ρ_0^* is the density expressed in units of solar masses per cubic parsec. In terms of the variable τ equation (1511) takes the simple form

$$\frac{\alpha_3}{\alpha_3(0)} = \frac{1}{(1 + \tau^2)^2} . \tag{1515}$$

Again, according to (1484), the foregoing relation is equivalent to

$$\frac{\rho}{\rho(0)} = \frac{1}{(1 + \tau^2)^2} . \tag{1516}$$

This law for the variation of density with time corresponds to a rather rapid rate of evolution of the system, for

$$\rho = 0.01\rho_0 \qquad \text{at} \qquad \tau = 3 , \tag{1517}$$

and this, according to equation (1514), implies that in a time

$$\frac{2.18 \times 10^7}{\sqrt{\rho_0^*}} \text{ years} \tag{1518}$$

the density decreases by a factor of 100 (see also Fig. 7). If $\rho_0^* \simeq 0.01$, (1518) provides a time scale of the order of 2×10^8 years.

However, in applying the formula (1516) it should be remembered that this law for the variation of density with time has been derived under circumstances which are valid, strictly speaking, only for "infinitely" (i.e., extremely!) flat systems,[65] and consequently the values of ρ_0^* appropriate for our present considerations should probably be of the order of 10^{-4}, in which case (1518) provides a time scale of the order of 3×10^9 years.

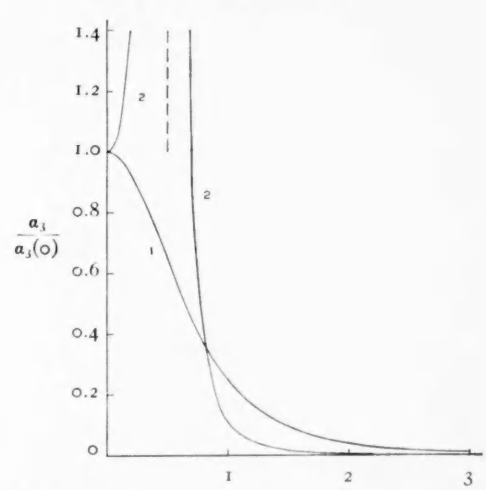

FIG. 7.—The evolution of spheroidal systems. Curve *1* illustrates the variation of density with time of an "infinitely" flat system, and curve *2* refers to the homologous evolution of a system with $\alpha_3/\alpha_1 = 0.8 =$ constant.

Passing on to a consideration of "moderately" flat systems, we first remark that the results obtained for "infinitely" flat systems cannot be extended uniquely to include the less extreme cases. This can be seen as follows: Expanding the various quantities occurring in equation (1486) in powers of α_1/α_3 and retaining only terms up to the first order of smallness, we obtain

$$3\,\frac{d^2Q_0}{dt^2} - \frac{1}{Q_0^3} + \frac{1}{4}\left(3\,\frac{d^2Q_1}{dt^2} + \frac{15Q_1}{Q_0^4}\right) = 0\,, \qquad (1519)$$

[65] It should be noted in this connection that in deriving Eq. (1503) from Eq. (1486) we have neglected $4\alpha_1$, as compared to α_3. For our own galactic system a/c is probably of the order of 15, and, according to (1485), $\alpha_3/\alpha_1 \simeq 20$; thus, in this case the neglect of $4\alpha_1$, as compared to α_3, is quite unsatisfactory.

where

$$Q_0 = \frac{1}{\alpha_3^{1/4}}\,; \qquad Q_1 = \frac{\alpha_1}{\alpha_3}\,\frac{1}{\alpha_3^{1/4}} = \frac{\alpha_1}{\alpha_3}\,Q_0\,. \tag{1520}$$

Consequently, Q_1 is a quantity of the first order of smallness, as compared to Q_0. If we neglect Q_1 altogether, equation (1519) reduces to (1503), which is the differential equation for an "infinitely" flat system. On the other hand, if Q_1 is a quantity of the first order of smallness, Q_0, strictly speaking, deviates from the solution (1508)

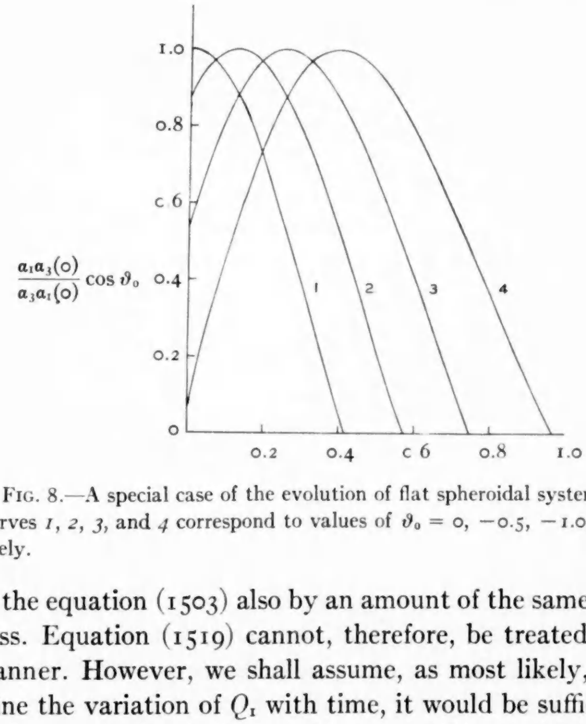

FIG. 8.—A special case of the evolution of flat spheroidal systems (see Eq. [1534]). Curves *1*, *2*, *3*, and *4* correspond to values of $\vartheta_0 = 0, -0.5, -1.0$, and -1.5, respectively.

of the equation (1503) also by an amount of the same order of smallness. Equation (1519) cannot, therefore, be treated in any unique manner. However, we shall assume, as most likely, that to determine the variation of Q_1 with time, it would be sufficient to use for Q_0 the expression derived in the "zeroth" approximation. In other words, we shall assume that Q_0 still satisfies the differential equation (1503). Under these circumstances Q_1 satisfies the differential equation

$$3\,\frac{d^2Q_1}{dt^2} + 15\,\frac{Q_1}{Q_0^4} = 0\,, \tag{1521}$$

where, according to equations (1508) and (1510),

$$Q_0^2 = \frac{1}{\sqrt{\alpha_3(0)}}\left[1 + \frac{\alpha_3(0)}{3}(t - t_0)^2\right]. \tag{1522}$$

Introducing equation (1522) in (1521), we have

$$3\frac{d^2Q_1}{dt^2} + \frac{15\alpha_3(0)}{\left[1 + \frac{\alpha_3(0)}{3}(t - t_0)^2\right]^2}Q_1 = 0. \tag{1523}$$

In terms of the variable τ (cf. Eq. [1512]) the foregoing equation reduces to

$$\frac{d^2Q_1}{d\tau^2} + \frac{15Q_1}{(1 + \tau^2)^2} = 0. \tag{1524}$$

Instead of τ, introduce the new variable ϑ, defined by

$$\tau = \tan\vartheta. \tag{1525}$$

Equation (1524) now transforms to

$$\frac{d^2Q_1}{d\vartheta^2} - 2\tan\vartheta\frac{dQ_1}{d\vartheta} + 15Q_1 = 0; \tag{1526}$$

this equation is readily seen to be equivalent to

$$\frac{d^2}{d\vartheta^2}(Q_1\cos\vartheta) + 16Q_1\cos\vartheta = 0. \tag{1527}$$

Hence,

$$Q_1\cos\vartheta = Q_{10}\cos(4\vartheta + \vartheta_0), \tag{1528}$$

where Q_{10} and ϑ_0 are two constants of integration. Returning to the variable τ, we have

$$Q_1 = Q_{10}\frac{\cos(4\tan^{-1}\tau + \vartheta_0)}{\cos(\tan^{-1}\tau)}. \tag{1529}$$

Since

$$\cos(\tan^{-1}\tau) \equiv \frac{1}{\sqrt{1 + \tau^2}}, \tag{1530}$$

we can re-write (1529) more conveniently as

$$Q_1 = Q_{10}\sqrt{1+\tau^2}\cos\left(4\tan^{-1}\tau+\vartheta_0\right) . \tag{1531}$$

Combining equations (1515), (1520), and (1531), we have

$$\frac{a_1}{a_3} = Q_{10}[a_3(0)]^{1/4}\cos\left(4\tan^{-1}\tau+\vartheta_0\right) . \tag{1532}$$

According to this equation,

$$\left(\frac{a_1}{a_3}\right)_{\tau=0} = Q_{10}[a_3(0)]^{1/4}\cos\vartheta_0 . \tag{1533}$$

Consequently, we can re-write equation (1532) in the form

$$\frac{a_1}{a_3} = \left(\frac{a_1}{a_3}\right)_{\tau=0}\sec\vartheta_0\cos\left(4\tan^{-1}\tau+\vartheta_0\right) . \tag{1534}$$

From (1534) it follows that

$$a_1 = 0 \qquad \text{when} \qquad 4\tan^{-1}\tau+\vartheta_0 = \pm\frac{\pi}{2} . \tag{1535}$$

Hence, for a_1 to be positive

$$-\frac{\pi}{2} \leqslant 4\tan^{-1}\tau+\vartheta_0 \leqslant \frac{\pi}{2} . \tag{1536}$$

In Figure 8 we have illustrated the nature of the dependence of a_1/a_3 on time. It should be emphasized, however, in this connection that in considering equation (1534) (and Fig. 8) we should remember that the variation of a_1/a_3 according to (1534) is not uniquely predicted by our theory. For, in deriving (1534) we have assumed that, as far as the variation of a_1/a_3 is concerned, it would be sufficient to use for a_3 the expression found valid in the limit $a_1/a_3 = 0$; but clearly this is not strictly true. On the other hand, the present discussion does confirm our earlier result (cf. Eqs. [1516] and [1517]) that within the framework of our fundamental postulates the evolution of very flat spheroidal systems is quite rapid. We shall later return to the cosmological significance of this result (cf. [iv], below).

iii) *The homologous evolution of spheroidal systems.*—We shall say that a spheroidal system evolves homologously if the ratio of the axes, κ, of the spheroid remains constant. According to the equations (1481), the constancy of κ implies the constancy of $\mathfrak{a}_1/\mathfrak{a}_3$. Let

$$\frac{\mathfrak{a}_3}{\mathfrak{a}_1} = k = \text{constant} . \tag{1537}$$

Equation (1486) now takes the form

$$3 \frac{d^2 Q}{dt^2} = \left(1 - \frac{4}{k}\right) \frac{1}{Q^3} , \tag{1538}$$

where

$$Q^4 = \mathfrak{a}_3^{-1} . \tag{1539}$$

The differential equation (1538) is of the same form as (1503) and can be solved by similar methods. We find (cf. Eq. [1511])

$$\mathfrak{a}_3 = \frac{\mathfrak{a}_3(0)}{\left[1 + \left(1 - \frac{4}{k}\right)\tau^2\right]^2} ; \qquad \mathfrak{a}_1 = \frac{\mathfrak{a}_1(0)}{\left[1 + \left(1 - \frac{4}{k}\right)\tau^2\right]^2} , \tag{1540}$$

where (as in Eq. [1512])

$$\tau = \sqrt{\frac{\mathfrak{a}_3(0)}{3}}\, (t - t_0) , \tag{1541}$$

and $\mathfrak{a}_1(0)$ and $\mathfrak{a}_3(0)$ are the values of $\mathfrak{a}_1$ and $\mathfrak{a}_3$ at time $t = t_0$.

From (1540) it immediately follows that $k = 4$ plays a critical role. Thus, for $k < 4$ the solution for $\mathfrak{a}_3$ has a singularity at

$$\tau = \sqrt{\frac{k}{4 - k}} ; \tag{1542}$$

but for $k > 4$ the solution has no singularity and is a monotonic decreasing function for $\tau \geqslant 0$. The case $k = 4$ itself is a singular one. The solution (1540) now shows that, consistent with the boundary condition $\dot{\mathfrak{a}}_3 = 0$ at $\tau = 0$, there can be no evolution along the line $\mathfrak{a}_3 = 4\mathfrak{a}_1$.[66] This is, of course, consistent with our results in (i)

[66] If $k = 4$, Eq. (1538) reduces to $\ddot{Q} = 0$. Hence,

$$\mathfrak{a}_1 \propto \mathfrak{a}_3 \propto (t - t_0)^{-4} \qquad (k = 4) . \tag{1540'}$$

above, since, according to (1499), $\nu_0 = 0$ and equation (1498) now require that α_1 and α_3 are constants.

iv) *General remarks on the evolution of elliptical nebulae.*—From our discussion of the differential equation (1486) the following facts have emerged: (1) Spheroidal systems E7.07 (i.e., systems the equilibrium states of which are represented by points on the line $\alpha_3 = 4\alpha_1$) are characterized by stable, as well as unstable, modes of vibration. Further, these systems are stationary for homologous evolution. (2) "Infinitely flat" systems evolve very rapidly. (3) Systems with ellipticities less than 0.707 are unstable against homologous evolution.

Now, it has been suggested (Part VII, § 37) that the existence of the critical case $\alpha_3 = 4\alpha_1$ may in some way be connected with the known upper limit, E7, to the sequence of elliptical nebulae. Before we examine this suggestion more closely from the point of view of the conclusions we have now reached, we shall first briefly state the position currently adopted.[67]

It is believed that the sequence of the elliptical nebulae stops at E7 and that at this point a "branching" occurs, the two branches being the sequence of the normal spirals, S, and the sequence of the barred spirals, SB, respectively. It is further generally supposed that the junction between the three sequences of nebulae is represented by a "more or less hypothetical class SO." According to Hubble, "the suggestion of a cataclysmic action at this critical point in the evolutional development of nebulae is rather pronounced."

In discussions such as those outlined above, the relative frequencies of the different types of nebulae are generally overlooked, and in the opinion of the writer this is particularly serious in considerations relating to the evolution of elliptical nebulae. Thus, it is known that "globular nebulae are relatively rare, as compared to lenticular systems [E7], and that numbers increase along the sequence with increasing ellipticity." This last-mentioned fact appears to be particularly curious, as it indicates a considerable abundance of the objects E7, i.e., precisely those which are believed to break up

[67] The writer's source of information on these and related matters has been Hubble's *The Realm of the Nebulae* (cf. n. 64, above). The quotations in the following paragraphs are from this book (chap. ii; see particularly pp. 45–46 and 55–56).

"cataclysmically." It appears that this relatively high abundance of the E7's is possible only if the intrinsic rate of evolution[68] of the elliptical nebulae decreases with increasing ellipticity. Now, this is exactly what our theory indicates. Indeed, for homologous evolution the E7's are stationary. Further, according to our stability considerations, the E7's are characterized by stable, as well as unstable, modes of vibration. It should be noticed in this connection that the unstable modes occur only in the case when $\mathfrak{a}_1$ and $\mathfrak{a}_3$ oscillate in phase and with the same frequency. It is therefore probable that the more likely mode of vibration is that described by Lissajous' curves (Fig. 6). We may thus expect that the instability which exists in the other "synchronized" mode of vibration may be only "mildly" operative. This would then mean that the systems E7 have a relatively slower rate of evolution than either the more or the less flattened systems. The point of view suggested here seems to be in general agreement with the facts of observations.

It may seem somewhat surprising that the case $\mathfrak{a}_3 = 4\mathfrak{a}_1$ should play this critical role. But this is probably connected with the following circumstance: Consider the equations of motion in the potential field

$$\mathfrak{V} = \tfrac{1}{2}[\mathfrak{a}_0(x^2 + y^2) + 4\mathfrak{a}_0 z^2] + \text{constant} , \tag{1543}$$

where $\mathfrak{a}_0$ is assumed to be a constant. We have

$$\frac{d^2x}{dt^2} = -\mathfrak{a}_0 x \; ; \qquad \frac{d^2y}{dt^2} = -\mathfrak{a}_0 y \; ; \qquad \frac{d^2z}{dt^2} = -4\mathfrak{a}_0 z \, . \tag{1544}$$

The solutions of these equations are

$$\left.\begin{aligned} x &= x_0 \cos \sqrt{\mathfrak{a}_0}\,(t - t_1) \, , \\ y &= y_0 \cos \sqrt{\mathfrak{a}_0}\,(t - t_2) \, , \\ z &= z_0 \cos 2\sqrt{\mathfrak{a}_0}\,(t - t_3) \, , \end{aligned}\right\} \tag{1545}$$

where x_0, y_0, z_0, t_1, t_2, and t_3 are the six constants of integration. According to the solution (1545), the frequency of oscillation in the z-direction is twice that in the x- or the y-direction. In other words, when $\mathfrak{a}_3 = 4\mathfrak{a}_1$, we have a case of *resonance*. It is to this resonance

[68] I.e., apart from the effects of disturbances.

that we are to attribute a "stabilizing" effect responsible for the relatively high abundance of the E7's. It is difficult to be more definite on this point without further investigation. But the suggestion is that the effects of a given disturbance are more rapidly dissipated when there is resonance than when there is not. This is clearly a matter of importance and will be examined on a future occasion.

In addition to the effects we have considered, there are, of course, other factors which are present. Thus, according to Lindblad's well-known work,[69] circular orbits at the periphery of a homogeneous spheroid with an eccentricity greater than 0.834 are unstable. But the important thing to which we wish to draw attention here is that, in addition to disrupting forces which undoubtedly exist in systems of high eccentricities, we may also expect certain stabilizing effects at some ellipticity near that of the E7's.

68. *Ellipsoidal systems.*—For general ellipsoidal systems we have

$$\alpha \neq \alpha_2 \neq \alpha_3 \,. \tag{1546}$$

We now proceed to a consideration of the fourteen systems of equations obtained in § 62.

From the systems I and II it readily follows that

$$f_{21} = g_{21} = h_{21} = f_{40} = g_{40} = h_{40} = 0 \,. \tag{1547}$$

Consider next the system III. As we have already shown (see pp. 537–538), only three of the six equations (ii) and (iii) are independent, and we can choose the first two equations of (ii) and (1411) as our three independent equations. On some further reductions involving the use of (i) these three equations become

$$\left.\begin{aligned} &3\,\frac{d^2 f_{11}}{dt^2} + f_{11}(3\alpha_2 - \alpha_1 + \alpha_3) + 2g_{11}(\alpha_2 - \alpha_3) = 0\,, \\ &3\,\frac{d^2 g_{11}}{dt^2} + g_{11}(3\alpha_1 - \alpha_2 + \alpha_3) + 2f_{11}(\alpha_1 - \alpha_3) = 0\,, \\ &4(\alpha_1 - \alpha_3)\,\frac{df_{11}}{dt} + f_{11}\,\frac{d}{dt}\,(\alpha_1 - \alpha_3) + 4(\alpha_2 - \alpha_3)\,\frac{dg_{11}}{dt} \\ &\qquad\qquad + g_{11}\,\frac{d}{dt}\,(\alpha_2 - \alpha_3) = 0\,. \end{aligned}\right\} \tag{1548}$$

[69] See, e.g., *Ap. J.*, **92**, 1, 1940.

The foregoing equations have to be further supplemented by the relation

$$f_{11} + g_{11} + h_{11} = 0 . \tag{1549}$$

The six systems IV–IX can be treated by methods similar to those we have used in our discussion of the systems IV, V, VII, and VIII in the spheroidal case (see pp. 539–540). We now find

$$f_{20} = \frac{\text{constant}}{|a_2 - a_3|^{1/4}}, \quad \text{IV}_1 ; \qquad 3\frac{d^2}{dt^2}\frac{1}{|a_2 - a_3|^{1/4}} + \frac{4a_2 - a_3}{|a_2 - a_3|^{1/4}} = 0 , \quad \text{IV}_2$$

$$g_{20} = \frac{\text{constant}}{|a_3 - a_1|^{1/4}}, \quad \text{V}_1 ; \qquad 3\frac{d^2}{dt^2}\frac{1}{|a_3 - a_1|^{1/4}} + \frac{4a_3 - a_1}{|a_3 - a_1|^{1/4}} = 0 , \quad \text{V}_2$$

$$h_{20} = \frac{\text{constant}}{|a_1 - a_2|^{1/4}}, \quad \text{VI}_1 ; \qquad 3\frac{d^2}{dt^2}\frac{1}{|a_1 - a_2|^{1/4}} + \frac{4a_1 - a_2}{|a_1 - a_2|^{1/4}} = 0 , \quad \text{VI}_2$$

$$f_{30} = \frac{\text{constant}}{|a_2 - a_3|^{1/4}}, \quad \text{VII}_1 ; \qquad 3\frac{d^2}{dt^2}\frac{1}{|a_2 - a_3|^{1/4}} + \frac{4a_3 - a_2}{|a_2 - a_3|^{1/4}} = 0 , \quad \text{VII}_2$$

$$g_{30} = \frac{\text{constant}}{|a_3 - a_1|^{1/4}}, \quad \text{VIII}_1 ; \qquad 3\frac{d^2}{dt^2}\frac{1}{|a_3 - a_1|^{1/4}} + \frac{4a_1 - a_3}{|a_3 - a_1|^{1/4}} = 0 , \quad \text{VIII}_2$$

$$h_{30} = \frac{\text{constant}}{|a_1 - a_2|^{1/4}}, \quad \text{IX}_1 ; \qquad 3\frac{d^2}{dt^2}\frac{1}{|a_1 - a_2|^{1/4}} + \frac{4a_2 - a_1}{|a_1 - a_2|^{1/4}} = 0 . \quad \text{IX}_2$$

It should be noted that the differential equations IV_2–IX_2 are obtained as eliminants and are valid only if the corresponding quantities f_{20}, g_{20}, h_{20}, f_{30}, g_{30}, and h_{30} do not vanish identically. Thus, if $f_{20} \equiv 0$, the differential equation IV_2 no longer follows, and similarly for the others. Comparing the differential equations IV_2 and VII_2, we at once see that these two equations cannot simultaneously be valid, since they are mutually inconsistent. Hence,

$$\text{either} \quad f_{20} \equiv 0 \quad \text{or} \quad f_{30} \equiv 0 . \tag{1550}$$

Similarly,

$$\left.\begin{aligned} &\text{either} \quad g_{20} \equiv 0 \quad \text{or} \quad g_{30} \equiv 0 , \\ &\text{either} \quad h_{20} \equiv 0 \quad \text{or} \quad h_{30} \equiv 0 . \end{aligned}\right\} \tag{1551}$$

Correspondingly,

$$\text{either} \quad \begin{pmatrix} \mathrm{IV}_2 \\ \mathrm{V}_2 \\ \mathrm{VI}_2 \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} \mathrm{VII}_2 \\ \mathrm{VIII}_2 \\ \mathrm{IX}_2 \end{pmatrix} \tag{1552}$$

is valid. From the foregoing it would follow that, consistent with equations (1550) and (1551), only three of the six quantities f_{20}, ... , h_{30} can be nonvanishing. We shall now show that, at most, only two of these six quantities can be nonvanishing and that, further, only two of the six differential equations IV_2, ... , IX_2 can simultaneously be valid:

Introduce the quantities Φ_1, Φ_2, and Φ_3, defined by

$$\Phi_1 = \sqrt[4]{\frac{3}{\alpha_3 - \alpha_2}}\,; \qquad \Phi_2 = \sqrt[4]{\frac{3}{\alpha_3 - \alpha_1}}\,; \qquad \Phi_3 = \sqrt[4]{\frac{3}{\alpha_2 - \alpha_1}}\,. \tag{1553}$$

In writing down the foregoing equations, we have assumed that

$$\alpha_3 > \alpha_2 > \alpha_1\,. \tag{1554}$$

We notice that the quantities Φ_1, Φ_2, and Φ_3 satisfy identically the relation

$$\frac{1}{\Phi_2^4} \equiv \frac{1}{\Phi_1^4} + \frac{1}{\Phi_3^4}\,. \tag{1555}$$

Consider now the differential equation IV_2. Since we can re-write it in either of the forms

$$\left.\begin{aligned} 3\alpha_3 - 4(\alpha_3 - \alpha_2) + 3(\alpha_3 - \alpha_2)^{1/4}\frac{d^2}{dt^2}\frac{1}{(\alpha_3 - \alpha_2)^{1/4}} &= 0\,, \\ 3\alpha_2 - (\alpha_3 - \alpha_2) + 3(\alpha_3 - \alpha_2)^{1/4}\frac{d^2}{dt^2}\frac{1}{(\alpha_3 - \alpha_2)^{1/4}} &= 0\,, \end{aligned}\right\} \tag{1556}$$

we readily obtain the relations

$$\alpha_3 = \frac{4}{\Phi_1^4} - \frac{\ddot{\Phi}_1}{\Phi_1}\,; \qquad \alpha_2 = \frac{1}{\Phi_1^4} - \frac{\ddot{\Phi}_1}{\Phi_1}\,. \tag{1557}$$

We can derive similar relations from the other equations, V_2–IX_2. Collecting all such relations, we have (cf. Eqs. [1550]–[1552])

$$
\left.\begin{array}{l}
\text{either } (1)\left\{\begin{aligned} \alpha_2 &= +\frac{1}{\Phi_1^4} - \frac{\breve{\Phi}_1}{\Phi_1}, \\ \alpha_3 &= +\frac{4}{\Phi_1^4} - \frac{\breve{\Phi}_1}{\Phi_1}, \end{aligned}\quad f_{20} \not\equiv 0\right\} \\
\qquad\qquad \text{or } (1')\left\{\begin{aligned} \alpha_2 &= -\frac{4}{\Phi_1^4} - \frac{\breve{\Phi}_1}{\Phi_1}, \\ \alpha_3 &= -\frac{1}{\Phi_1^4} - \frac{\breve{\Phi}_1}{\Phi_1}, \end{aligned}\quad f_{30} \not\equiv 0\right\} \\
\text{either } (2)\left\{\begin{aligned} \alpha_3 &= -\frac{1}{\Phi_2^4} - \frac{\breve{\Phi}_2}{\Phi_2}, \\ \alpha_1 &= -\frac{4}{\Phi_2^4} - \frac{\breve{\Phi}_2}{\Phi_2}, \end{aligned}\quad g_{20} \not\equiv 0\right\} \\
\qquad\qquad \text{or } (2')\left\{\begin{aligned} \alpha_3 &= +\frac{4}{\Phi_2^4} - \frac{\breve{\Phi}_2}{\Phi_2}, \\ \alpha_1 &= +\frac{1}{\Phi_2^4} - \frac{\breve{\Phi}_2}{\Phi_2}, \end{aligned}\quad g_{30} \not\equiv 0\right\} \\
\text{either } (3)\left\{\begin{aligned} \alpha_1 &= +\frac{1}{\Phi_3^4} - \frac{\breve{\Phi}_3}{\Phi_3}, \\ \alpha_2 &= +\frac{4}{\Phi_3^4} - \frac{\breve{\Phi}_3}{\Phi_3}, \end{aligned}\quad h_{20} \not\equiv 0\right\} \\
\qquad\qquad \text{or } (3')\left\{\begin{aligned} \alpha_1 &= -\frac{4}{\Phi_3^4} - \frac{\breve{\Phi}_3}{\Phi_3}, \\ \alpha_2 &= -\frac{1}{\Phi_3^4} - \frac{\breve{\Phi}_3}{\Phi_3}. \end{aligned}\quad h_{30} \not\equiv 0\right\}
\end{array}\right\} \qquad (1558)
$$

We shall now show that any permissible set of *three* from the foregoing six systems cannot be simultaneously valid. For, if not, suppose that (1), (2), and (3) are simultaneously valid. Then, according to these equations, we have

$$
\frac{1}{\Phi_3^4} - \frac{\breve{\Phi}_3}{\Phi_3} = -\frac{4}{\Phi_2^4} - \frac{\breve{\Phi}_2}{\Phi_2} \qquad (1559)
$$

and

$$\frac{4}{\Phi_3^4} - \frac{\ddot{\Phi}_3}{\Phi_3} = \frac{1}{\Phi_1^4} - \frac{\ddot{\Phi}_1}{\Phi_1} . \tag{1560}$$

We can now apply to the foregoing equations the theorems of § 63. From equations (1559) and (1560) we obtain (cf. Eq. [1367])

$$\left. \begin{aligned} (\Phi_3^2 \dot{u}_{23})^2 &= q_{23} + \frac{4}{u_{23}^2} - u_{23}^2 , \\ (\Phi_3^2 \dot{u}_{13})^2 &= q_{13} - \frac{1}{u_{13}^2} - 4u_{13}^2 , \end{aligned} \right\} \tag{1561}$$

where q_{23} and q_{13} are constants and

$$u_{23} = \frac{\Phi_2}{\Phi_3} ; \qquad u_{13} = \frac{\Phi_1}{\Phi_3} . \tag{1562}$$

Hence,

$$\left(\frac{\dot{u}_{23}}{\dot{u}_{13}}\right)^2 = \frac{q_{23} + \dfrac{4}{u_{23}^2} - u_{23}^2}{q_{13} - \dfrac{1}{u_{13}^2} - 4u_{13}^2} . \tag{1563}$$

On the other hand, according to equations (1555) and (1562), we have the identity

$$1 + \frac{1}{u_{13}^4} = \frac{1}{u_{23}^4} . \tag{1564}$$

The equations (1563) and (1564) are clearly incompatible, and we are thus led to a contradiction. This proves our statement that only two of the six differential equations $\text{IV}_2, \ldots, \text{IX}_2$ can be simultaneously valid. This, in turn, implies that only two of the six quantities $f_{20}, \ldots, h_{30}$ can be simultaneously nonvanishing. Depending on which two we choose (consistent with Eqs. [1550] and [1551]), we have the following twelve possible combinations from the systems (1), (2), (3), (1′), (2′), and (3′) (Eq. [1558]):

$$\left. \begin{matrix} (1, 2) ; & (2, 3) ; & (3, 1) ; & (1', 2') ; & (2', 3') ; & (3', 1') , \\ (1, 2') ; & (2, 3') ; & (3, 1') ; & (1', 2) ; & (2', 3) ; & (3', 1) . \end{matrix} \right\} \tag{1565}$$

We shall later return to these possible combinations.

Consider next the systems X–XII. These can be treated by meth-

ods similar to those used in our discussion of the systems X and XI in the spheroidal case (see pp. 540–542). We find

$$\left.\begin{aligned}&\frac{df_{10}}{dt} = \beta_1 + \gamma_1\,; \qquad \frac{d}{dt}(\beta_1 - \gamma_1) = (\alpha_2 - \alpha_3)f_{10}\,,\\&(\alpha_2 + \alpha_3)f_{10}^2 + f_{10}\frac{d^2 f_{10}}{dt^2} = \text{constant} + 2\beta_1\gamma_1\,,\end{aligned}\right\} \tag{1566}$$

$$\left.\begin{aligned}&\frac{dg_{10}}{dt} = \beta_2 + \gamma_2\,; \qquad \frac{d}{dt}(\beta_2 - \gamma_2) = (\alpha_3 - \alpha_1)g_{10}\,,\\&(\alpha_3 + \alpha_1)g_{10}^2 + g_{10}\frac{d^2 g_{10}}{dt^2} = \text{constant} + 2\beta_2\gamma_2\,,\end{aligned}\right\} \tag{1567}$$

and

$$\left.\begin{aligned}&\frac{dh_{10}}{dt} = \beta_3 + \gamma_3\,; \qquad \frac{d}{dt}(\beta_3 - \gamma_3) = (\alpha_1 - \alpha_2)h_{10}\,,\\&(\alpha_1 + \alpha_2)h_{10}^2 + h_{10}\frac{d^2 h_{10}}{dt^2} = \text{constant} + 2\beta_3\gamma_3\,.\end{aligned}\right\} \tag{1568}$$

And finally we have the systems XIII and XIV. According to these equations, we have

$$\left.\begin{aligned}\alpha_1 &= -\frac{\ddot{\delta}_1}{\delta_1} = \frac{q_1}{a_0^2} - \frac{1}{\sqrt{a_0}}\frac{d^2}{dt^2}\sqrt{a_0}\,,\\\alpha_2 &= -\frac{\ddot{\delta}_2}{\delta_2} = \frac{q_2}{b_0^2} - \frac{1}{\sqrt{b_0}}\frac{d^2}{dt^2}\sqrt{b_0}\,,\\\alpha_3 &= -\frac{\ddot{\delta}_3}{\delta_3} = \frac{q_3}{c_0^2} - \frac{1}{\sqrt{c_0}}\frac{d^2}{dt^2}\sqrt{c_0}\,,\end{aligned}\right\} \tag{1569}$$

where q_1, q_2, and q_3 are constants. We can now apply the results of § 63 to express a_0, b_0, and c_0 in terms of δ_1, δ_2, and δ_3, respectively. However, it would be more convenient to express all the quantities in terms of the Φ's according to equation (1558), but this would depend upon which of the combinations (1565) we choose.

a) The case when α_1, α_2, and α_3 are constants.—As in the spheroidal case (§ 66) the explicit expressions for the various quantities can be written down when α_1, α_2, and α_3 are constants. Thus, we have:

(i)

$$f_{21} = g_{21} = h_{21} = f_{40} = g_{40} = h_{40} = 0\,. \tag{1570}$$

(ii) From the equations (1548) it readily follows that f_{11}, g_{11}, and h_{11} are independent of time. The further discussion of these equations, together with the equations IV_1–IX_2, leads to the same situation as has already been encountered in the steady-state problem (Part VIII, § 37, pp. 147–150). In particular we have to distinguish between the four cases (888). It will be noticed that for the critical cases which now arise (e.g., $\alpha_1 : \alpha_2 : \alpha_3 = 1 : 4 : 16$ and $1 : 4 : 9$) the frequencies of oscillations of the individual stars (cf. Eqs. [1545]) along the x, y, and z directions are in simple ratios (e.g., $1 : 2 : 4$ and $1 : 2 : 3$). Thus these critical cases again correspond to cases of resonance.

(iii) The solutions for β_1, γ_1, etc., according to the systems X–XII (§ 62) are readily obtained. We find (cf. Eqs. [1475], [1477], and [1478]) that

$$\left.\begin{aligned}
\beta_1 &= \sqrt{\alpha_3}[D_{11} \sin(\sqrt{\alpha_3} + \sqrt{\alpha_2})t + D_{12} \sin(\sqrt{\alpha_3} - \sqrt{\alpha_2})t] \,, \\
\gamma_1 &= \sqrt{\alpha_2}[D_{11} \sin(\sqrt{\alpha_3} + \sqrt{\alpha_2})t - D_{12} \sin(\sqrt{\alpha_3} - \sqrt{\alpha_2})t] \,, \\
f_{10} &= - \quad [D_{11} \cos(\sqrt{\alpha_3} + \sqrt{\alpha_2})t + D_{12} \cos(\sqrt{\alpha_3} - \sqrt{\alpha_2})t] \,, \\
\beta_2 &= \sqrt{\alpha_1}[D_{21} \sin(\sqrt{\alpha_1} + \sqrt{\alpha_3})t + D_{22} \sin(\sqrt{\alpha_1} - \sqrt{\alpha_3})t] \,, \\
\gamma_2 &= \sqrt{\alpha_3}[D_{21} \sin(\sqrt{\alpha_1} + \sqrt{\alpha_3})t - D_{22} \sin(\sqrt{\alpha_1} - \sqrt{\alpha_3})t] \,, \\
g_{10} &= - \quad [D_{21} \cos(\sqrt{\alpha_1} + \sqrt{\alpha_3})t + D_{22} \cos(\sqrt{\alpha_1} - \sqrt{\alpha_3})t] \,, \\
\beta_3 &= \sqrt{\alpha_2}[D_{31} \sin(\sqrt{\alpha_2} + \sqrt{\alpha_1})t + D_{32} \sin(\sqrt{\alpha_2} - \sqrt{\alpha_1})t] \,, \\
\gamma_3 &= \sqrt{\alpha_1}[D_{31} \sin(\sqrt{\alpha_2} + \sqrt{\alpha_1})t - D_{32} \sin(\sqrt{\alpha_2} - \sqrt{\alpha_1})t] \,, \\
h_{10} &= - \quad [D_{31} \cos(\sqrt{\alpha_2} + \sqrt{\alpha_1})t + D_{32} \cos(\sqrt{\alpha_2} - \sqrt{\alpha_1})t] \,,
\end{aligned}\right\} \quad (1571)$$

where $D_{11}, \ldots, D_{32}$ are constants of integration.

(iv) And, finally, according to equations (1569), we have

$$\left.\begin{aligned}
a_0 &= a_{00} + a_{01} \sin 2\sqrt{\alpha_1}(t - t_1) \,; & \delta_1 &= \delta_{10} \sin \sqrt{\alpha_1}(t - t_1') \,, \\
b_0 &= b_{00} + b_{01} \sin 2\sqrt{\alpha_2}(t - t_2) \,; & \delta_2 &= \delta_{20} \sin \sqrt{\alpha_2}(t - t_2') \,, \\
c_0 &= c_{00} + c_{01} \sin 2\sqrt{\alpha_3}(t - t_3) \,; & \delta_3 &= \delta_{30} \sin \sqrt{\alpha_3}(t - t_3') \,,
\end{aligned}\right\} \quad (1572)$$

where $a_{00}, \ldots, c_{01}$ and $t_1, \ldots, t_3'$ are all arbitrary constants.

Thus, in general, the solution for the coefficients of the velocity ellipsoid and the motions of the local centroids involves twenty-one

constants of integration, and in the special cases (Eq. [888]) we have one or two additional constants.

b) The general case.—We have already seen how a consideration of the equations (1558) leads us to distinguish between the twelve cases (1565). We shall now show that in each of these twelve cases we have a relation which enables us to express one of the Φ's in terms of another. Thus, let us consider the systems (1) and (2) of equation (1558). According to these equations,

$$\alpha_3 = \frac{4}{\Phi_1^4} - \frac{\ddot{\Phi}_1}{\Phi_1} = -\frac{1}{\Phi_2^4} - \frac{\ddot{\Phi}_2}{\Phi_2}. \tag{1573}$$

We can now apply to the foregoing equation the results of § 63. We have (cf. Eqs. [1372] and [1373])

$$\left.\begin{aligned} \pm 2\int \frac{dt}{\Phi_1^2} &= \tfrac{1}{2}\cos^{-1}\frac{q - 8u_{21}^2}{\sqrt{q^2+16}}, \\ \mp 2\int \frac{dt}{\Phi_2^2} &= \cosh^{-1}\frac{q + 2u_{12}^2}{\sqrt{q^2+16}}, \end{aligned}\right\} \tag{1574}$$

where

$$u_{21} = \frac{\Phi_2}{\Phi_1}; \qquad u_{12} = \frac{\Phi_1}{\Phi_2} \tag{1575}$$

and q is an arbitrary constant. The relations (1574) can be re-written as (cf. Eq. [1553])

$$\left.\begin{aligned} q - 8\sqrt{\frac{\alpha_3 - \alpha_2}{\alpha_3 - \alpha_1}} &= \sqrt{q^2+16}\cos\frac{4}{\sqrt{3}}\int^t \sqrt{\alpha_3 - \alpha_2}\,dt, \\ q + 2\sqrt{\frac{\alpha_3 - \alpha_1}{\alpha_3 - \alpha_2}} &= \sqrt{q^2+16}\cosh\frac{2}{\sqrt{3}}\int^t \sqrt{\alpha_3 - \alpha_1}\,dt. \end{aligned}\right\} \tag{1576}$$

Since α_1, α_2, and α_3 are all determined in terms of Φ_1 and Φ_2 and since we now have the relation (1574) between them, it follows that in the specification of α_1, α_2, and α_3 there is the arbitrariness of only a single unknown function, namely, Φ_1 or Φ_2. In terms of this one arbitrary function all the other quantities can be expressed.

The consideration of the other combinations (1565) leads to relations similar to (1574). Collecting all such relations, we have

$$\pm 2\int\frac{dt}{\Phi_1^2} = \tfrac{1}{2}\cos^{-1}\frac{q-8u_{21}^2}{\sqrt{q^2+16}}; \qquad \mp 2\int\frac{dt}{\Phi_2^2} = \cosh^{-1}\frac{q+2u_{12}^2}{\sqrt{q^2+16}}, \qquad (1,\ 2)$$

$$\pm 2\int\frac{dt}{\Phi_2^2} = \tfrac{1}{2}\cosh^{-1}\frac{q+8u_{32}^2}{\sqrt{q^2+16}}; \qquad \mp 2\int\frac{dt}{\Phi_3^2} = \cos^{-1}\frac{q-2u_{23}^2}{\sqrt{q^2+16}}, \qquad (2,\ 3)$$

$$\pm 2\int\frac{dt}{\Phi_3^2} = \tfrac{1}{2}\cos^{-1}\frac{q-8u_{13}^2}{\sqrt{q^2-16}}; \qquad \mp 2\int\frac{dt}{\Phi_1^2} = \cos^{-1}\frac{q-2u_{31}^2}{\sqrt{q^2-16}}, \qquad (3,\ 1)$$

$$\pm 2\int\frac{dt}{\Phi_1^2} = \cosh^{-1}\frac{q+2u_{21}^2}{\sqrt{q^2+16}}; \qquad \mp 2\int\frac{dt}{\Phi_2^2} = \tfrac{1}{2}\cos^{-1}\frac{q-8u_{12}^2}{\sqrt{q^2+16}}, \qquad (1',\ 2')$$

$$\pm 2\int\frac{dt}{\Phi_2^2} = \cos^{-1}\frac{q-2u_{32}^2}{\sqrt{q^2+16}}; \qquad \mp 2\int\frac{dt}{\Phi_3^2} = \tfrac{1}{2}\cosh^{-1}\frac{q+8u_{23}^2}{\sqrt{q^2+16}}, \qquad (2',\ 3')$$

$$\left.\begin{array}{l} \pm 2\displaystyle\int\frac{dt}{\Phi_3^2} = \sinh^{-1}\frac{q+2u_{13}^2}{\sqrt{16-q^2}} \quad \text{or} \quad \cosh^{-1}\frac{q+2u_{13}^2}{\sqrt{q^2-16}} \\ \mp 2\displaystyle\int\frac{dt}{\Phi_1^2} = \tfrac{1}{2}\sinh^{-1}\frac{q+8u_{31}^2}{\sqrt{16-q^2}} \quad \text{or} \quad \tfrac{1}{2}\cosh^{-1}\frac{q+8u_{31}^2}{\sqrt{q^2-16}} \end{array}\right\}, \qquad (3',\ 1')$$

$$\pm 2\int\frac{dt}{\Phi_1^2} = \tfrac{1}{2}\cos^{-1}\frac{q-8u_{21}^2}{\sqrt{q^2-64}}; \qquad \mp 2\int\frac{dt}{\Phi_2^2} = \tfrac{1}{2}\cos^{-1}\frac{q-8u_{12}^2}{\sqrt{q^2-64}}, \qquad (1,\ 2') \qquad (1577)$$

$$\left.\begin{array}{l} \pm 2\displaystyle\int\frac{dt}{\Phi_2^2} = \tfrac{1}{2}\sinh^{-1}\frac{q+8u_{32}^2}{\sqrt{64-q^2}} \quad \text{or} \quad \tfrac{1}{2}\cosh^{-1}\frac{q+8u_{32}^2}{\sqrt{q^2-64}} \\ \mp 2\displaystyle\int\frac{dt}{\Phi_3^2} = \tfrac{1}{2}\sinh^{-1}\frac{q+8u_{23}^2}{\sqrt{64-q^2}} \quad \text{or} \quad \tfrac{1}{2}\cosh^{-1}\frac{q+8u_{23}^2}{\sqrt{q^2-64}} \end{array}\right\}, \qquad (2,\ 3')$$

$$\pm 2\int\frac{dt}{\Phi_3^2} = \tfrac{1}{2}\cos^{-1}\frac{q-8u_{13}^2}{\sqrt{q^2+64}}; \qquad \mp 2\int\frac{dt}{\Phi_1^2} = \tfrac{1}{2}\cosh^{-1}\frac{q+8u_{31}^2}{\sqrt{q^2+64}}, \qquad (3,\ 1')$$

$$\left.\begin{array}{l} \pm 2\displaystyle\int\frac{dt}{\Phi_1^2} = \sinh^{-1}\frac{q+2u_{21}^2}{\sqrt{4-q^2}} \quad \text{or} \quad \cosh^{-1}\frac{q+2u_{21}^2}{\sqrt{q^2-4}} \\ \mp 2\displaystyle\int\frac{dt}{\Phi_2^2} = \sinh^{-1}\frac{q+2u_{12}^2}{\sqrt{4-q^2}} \quad \text{or} \quad \cosh^{-1}\frac{q+2u_{12}^2}{\sqrt{q^2-4}} \end{array}\right\}, \qquad (1',\ 2)$$

$$\pm 2\int\frac{dt}{\Phi_2^2} = \cos^{-1}\frac{q-2u_{32}^2}{\sqrt{q^2-4}}; \qquad \mp 2\int\frac{dt}{\Phi_3^2} = \cos^{-1}\frac{q-2u_{23}^2}{\sqrt{q^2-4}}, \qquad (2',\ 3)$$

$$\pm 2\int\frac{dt}{\Phi_3^2} = \cosh^{-1}\frac{q+2u_{13}^2}{\sqrt{q^2+4}}; \qquad \mp 2\int\frac{dt}{\Phi_1^2} = \cos^{-1}\frac{q-2u_{31}^2}{\sqrt{q^2+4}}. \qquad (3,\ 1')$$

Again, combining the expressions for the α's in terms of the Φ's with the equations (1569), we can express each of the quantities δ_1, δ_2, δ_3, a_0, b_0, and c_0 in terms of the appropriate Φ's by transcendental relations of the forms (1380) or (1372).

Finally, it should be noticed that the elimination of f_{11} and g_{11} from the equations (1548) will lead to a further relation between the Φ's. We should therefore determine whether this new eliminant relation is consistent with (1577): if these two relations are not consistent, we should ignore one or the other by making the appropriate quantities vanish identically; if they are consistent, we may be able to solve for α_1, α_2, and α_3 uniquely. We shall not continue further with this inquiry here but shall postpone it to a further occasion.

XIII. SPIRAL STRUCTURE

69. *The formulation of a problem.*—In our discussion of stellar systems in nonsteady states (Parts X–XII) we have restricted ourselves to a consideration of such systems as have analogues in the steady-state theory. The problem now presents itself whether within the framework of the nonsteady-state theory we are not allowed types of stellar systems which have no strict counterparts in the steady-state theory. That such systems probably exist is indicated by the following consideration: Under steady-state circumstances the gravitational potential, $\mathfrak{V}$, satisfies the equation (cf. Part IV, Eq. [206])

$$\Delta \cdot \operatorname{grad} \mathfrak{V} = 0 , \tag{1578}$$

and the solution of this linear homogeneous partial differential equation shows that $\mathfrak{V}$ must be characterized by a helical symmetry (Part IV). In practice, however, we are restricted to consider only systems with axial symmetries.[70] On the other hand, in the nonsteady-state theory $\mathfrak{V}$ is not required to satisfy an equation of the form (1578); instead, we have three further integrability conditions (Eq. [960]). Consequently, with the greater freedom which this provides we may be able to discover types of stellar systems which have no strict analogues under steady-state circumstances. If such systems exist, it is, of course, important to isolate them. From a

[70] See Part XV (now under preparation).

practical point of view, the search for systems beyond the range of the steady-state theory insists itself, since our discussions of stellar systems in steady states and their analogues in nonsteady states have not so far disclosed any place in our theory where we may hope to find solutions for the more difficult problems of stellar dynamics, such as, for instance, the structure and the development of a spiral nebula. It is, of course, possible that we may not find solutions for these problems of "higher" stellar dynamics within the framework of our fundamental kinematical postulates I, II, and III (Part IX, pp. 445–446). However, the writer does not see any obvious reason why any of the three postulates should break down for the extragalactic nebulae. In any case, the problem of specifying the most general type of stellar system which is consistent with our kinematical postulates becomes a matter of some urgency.

At the outset we are faced with some ambiguity in interpreting the phrase "the most general type of stellar system." We can take this to mean one of two things: Either we may require flexibility in the specification of the coefficients of the velocity ellipsoid, or we may require flexibility in the specification of the potential and the density distributions. Because of the integrability conditions (Eqs. [958] and [960]), the two requirements are, to a certain extent, "complementary"[71] in character. Or, in other words, the greater the degree of arbitrariness we wish to have in the specification of the velocity ellipsoid, the less, in general, is the arbitrariness we are left with in the specification of the potential $\mathfrak{V}$, and conversely.[72] From the point of view of the considerations outlined in the preceding paragraph, we are primarily interested in the potential and the density distributions. Consequently, it would appear that we should impose the maximum restrictions on the form of the velocity ellipsoid. Since we cannot restrict the form of the velocity ellipsoid to a greater extent than to suppose that it is a sphere, the problem which suggests itself is the following:

What are the characteristics of a stellar system which is described by a spherical distribution of the residual velocities?

[71] "Complementary" in the sense of Bohr.

[72] This is clearly illustrated in our discussion of the two-dimensional problem under steady-state conditions (Part II, § 9; see particularly the remarks following Eq. [85]).

It is to the solution of this problem that this part is devoted. As we shall see, the analysis discloses a class of stellar systems which is strongly suggestive of the spiral and other nebulae.

70. *The equations for a stellar system having a spherical distribution of the residual velocities.*—A spherical distribution of the residual velocities implies that the distribution function Ψ has the form

$$\Psi \equiv \Psi[a\{(U - U_0)^2 + (V - V_0)^2 + (W - W_0)^2\} + \sigma] , \quad (1579)$$

where a, U_0, V_0, W_0, and σ are, in the first instance, arbitrary functions of x, y, z, and t. In equation (1579), $a^{-1/2}$ is proportional to the dispersion of the residual velocities, U_0, V_0, and W_0 are the components of the motion of the local centroid in a fixed Cartesian frame of reference, and σ is the density function.

Now, according to our general solution for the coefficients of the velocity ellipsoid (Eqs. [941]), it follows that

$$a \equiv \kappa(t) , \quad (1580)$$

where, as the notation implies, κ is a function of time only. Further, according to equations (933), (949), and (953), we have

$$\left.\begin{aligned} \Delta_1 &= \kappa U_0 = \frac{1}{2}\frac{d\kappa}{dt}x + \beta_3 y - \beta_2 z + \delta_1 , \\ \Delta_2 &= \kappa V_0 = \frac{1}{2}\frac{d\kappa}{dt}y + \beta_1 z - \beta_3 x + \delta_2 , \\ \Delta_3 &= \kappa W_0 = \frac{1}{2}\frac{d\kappa}{dt}z + \beta_2 x - \beta_1 y + \delta_3 , \end{aligned}\right\} \quad (1581)$$

where β_1, β_2, β_3, δ_1, δ_2, and δ_3 are all functions of time only. We can write (1581) in the form of a single vector equation, as

$$\boldsymbol{\Delta} = \frac{1}{2}\frac{d\kappa}{dt}\boldsymbol{r} + \boldsymbol{r} \times \boldsymbol{\beta} + \boldsymbol{\delta} , \quad (1582)$$

where $\boldsymbol{\Delta}$, $\boldsymbol{\beta}$, and $\boldsymbol{\delta}$ denote the vectors $(\Delta_1, \Delta_2, \Delta_3)$, $(\beta_1, \beta_2, \beta_3)$, and $(\delta_1\ \delta_2, \delta_3)$, respectively. It may be further noted that $\boldsymbol{\Delta}$ is proportional to the vector (U_0, V_0, W_0) defining the motion of the local

centroid. Finally, our compatibility relations are (Eqs. [956] and [957])

$$\kappa \operatorname{grad} \mathfrak{V} + \frac{\partial \boldsymbol{\Delta}}{\partial t} = -\tfrac{1}{2} \operatorname{grad} \chi \tag{1583}$$

and

$$\boldsymbol{\Delta} \cdot \operatorname{grad} \mathfrak{V} = \frac{1}{2} \frac{\partial \chi}{\partial t}, \tag{1584}$$

where χ stands for (cf. Eqs. [934] and [935])

$$-\chi = \kappa(U_0^2 + V_0^2 + W_0^2) + \sigma = Q_0 + \sigma. \tag{1585}$$

Taking the curl of equation (1583), we obtain

$$\frac{\partial}{\partial t} (\operatorname{curl} \boldsymbol{\Delta}) = 0. \tag{1586}$$

But, according to equation (1582),

$$\operatorname{curl} \boldsymbol{\Delta} = -2\boldsymbol{\beta}. \tag{1587}$$

Hence,

$$\frac{d\boldsymbol{\beta}}{dt} = 0; \tag{1588}$$

in other words, $\boldsymbol{\beta}$ *is a constant vector.*

The three terms which occur in the expression for $\boldsymbol{\Delta}$ have now simple interpretations. The first term corresponds to a radial expansion (or contraction) of amount

$$\kappa P = \frac{1}{2} \frac{d\kappa}{dt} \boldsymbol{r} \cdot \frac{\boldsymbol{r}}{r} = \frac{1}{2} \frac{d\kappa}{dt} r; \tag{1589}$$

the second term corresponds to a rotation Θ_0 about an axis through the origin and parallel to the $\boldsymbol{\beta}$-direction given by

$$\kappa \Theta_0 = -|\boldsymbol{\beta}| \bar{\omega}, \tag{1590}$$

where $\bar{\omega}$ is the distance measured from the axis of rotation; and, finally, the third term corresponds to a general translation. We have

thus shown that *for a stellar system with a spherical distribution of velocities the motions of the local centroids can be expressed as the superposition of an expansion (or contraction) about a fixed origin and proportional to the distance from the origin, a rotation about a fixed axis through the same origin and proportional to the distance from the axis of rotation and a general translation.*

Consider next the integrability condition resulting from the equations (1583) and (1584). We have

$$\operatorname{grad}(\boldsymbol{\Delta}\cdot\operatorname{grad}\mathfrak{B})+\frac{\partial}{\partial t}(\kappa\operatorname{grad}\mathfrak{B})+\frac{\partial^2\boldsymbol{\Delta}}{\partial t^2}=0\,. \tag{1591}$$

Substituting for $\boldsymbol{\Delta}$ according to (1582) in the foregoing equation and remembering that $\boldsymbol{\beta}$ is a constant vector, we have

$$\operatorname{grad}(\boldsymbol{\Delta}\cdot\operatorname{grad}\mathfrak{B})+\frac{\partial}{\partial t}(\kappa\operatorname{grad}\mathfrak{B})+\frac{1}{2}\frac{d^3\kappa}{dt^3}\boldsymbol{r}+\frac{d^2\boldsymbol{\delta}}{dt^2}=0\,. \tag{1592}$$

It is readily verified that the equation (1592) is equivalent to

$$\operatorname{grad}\left\{\boldsymbol{\Delta}\cdot\operatorname{grad}\mathfrak{B}+\frac{\partial}{\partial t}(\kappa\mathfrak{B})+\frac{1}{4}\frac{d^3\kappa}{dt^3}r^2+\frac{d^2\boldsymbol{\delta}}{dt^2}\cdot\boldsymbol{r}\right\}=0\,. \tag{1593}$$

Hence,

$$\boldsymbol{\Delta}\cdot\operatorname{grad}\mathfrak{B}+\frac{\partial}{\partial t}(\kappa\mathfrak{B})+\frac{1}{4}\frac{d^3\kappa}{dt^3}r^2+\frac{d^2\boldsymbol{\delta}}{dt^2}\cdot\boldsymbol{r}=F(t)\,, \tag{1594}$$

where $F(t)$ is an arbitrary function of time only.

In considering the equation (1594), there is no loss of generality if we set $F(t)=0$ (cf. Part XI, § 60). For, if we write

$$\mathfrak{B}=\mathfrak{B}_1(x,y,z,t)+\frac{1}{\kappa}\int^t F(t)dt\,, \tag{1595}$$

it readily follows from (1594) that $\mathfrak{B}_1$ satisfies the differential equation

$$\boldsymbol{\Delta}\cdot\operatorname{grad}\mathfrak{B}+\frac{\partial}{\partial t}(\kappa\mathfrak{B})+\frac{1}{4}\frac{d^3\kappa}{dt^3}r^2+\frac{d^2\boldsymbol{\delta}}{dt^2}\cdot\boldsymbol{r}=0\,. \tag{1596}$$

Since the addition of an arbitrary function of time to the gravitational potential has no physical significance, we can simply ignore the additive term in (1595). Substituting for $\boldsymbol{\Delta}$ according to (1582), equation (1596) takes explicitly the form

$$\left.\begin{aligned}\left(\frac{1}{2}\frac{d\kappa}{dt}\boldsymbol{r}+\boldsymbol{r}\times\boldsymbol{\beta}+\boldsymbol{\delta}\right)\cdot\operatorname{grad}\mathfrak{B}+\frac{\partial}{\partial t}(\kappa\mathfrak{B})+\frac{1}{4}\frac{d^3\kappa}{dt^3}r^2\\ +\frac{d^2\boldsymbol{\delta}}{dt^2}\cdot\boldsymbol{r}=0\,.\end{aligned}\right\}\qquad(1597)$$

Equation (1597) is a linear nonhomogeneous partial differential equation for $\mathfrak{B}$, and it is on this equation that the solution to our problem hinges.

Equation (1597) will be considered in §§ 71–74, but meantime we shall find the solutions for χ and σ, supposing that the general solution of equation (1597) is known.

Explicitly, equation (1583) takes the form

$$-\tfrac{1}{2}\operatorname{grad}\chi=\kappa\operatorname{grad}\mathfrak{B}+\frac{1}{2}\frac{d^2\kappa}{dt^2}\boldsymbol{r}+\frac{d\boldsymbol{\delta}}{dt}\,;\qquad(1598)$$

or, solving directly for χ, we have

$$-\tfrac{1}{2}\chi=\kappa\mathfrak{B}+\frac{1}{4}\frac{d^2\kappa}{dt^2}r^2+\frac{d\boldsymbol{\delta}}{dt}\cdot\boldsymbol{r}-\tfrac{1}{2}\chi_0(t)\,,\qquad(1599)$$

where χ_0 is an arbitrary function of time only. We can suppose, without any loss of generality, that in (1599) $\mathfrak{B}$ denotes the general solution of (1597). For, since the general solution of equation (1597) differs from the general solution of equation (1594) only through an arbitrary function of time, we can allow for this by redefining χ_0; or, in other words, we can absorb the additive function of time in equation (1595) in our definition of χ_0.

From equation (1599) it follows that

$$-\frac{1}{2}\frac{\partial\chi}{\partial t}=\frac{\partial}{\partial t}(\kappa\mathfrak{B})+\frac{1}{4}\frac{d^3\kappa}{dt^3}r^2+\frac{d^2\boldsymbol{\delta}}{dt^2}\cdot\boldsymbol{r}-\frac{1}{2}\frac{d\chi_0}{dt}\,.\qquad(1600)$$

Combining equations (1584) and (1600), we have

$$\frac{1}{2}\frac{d\chi_0}{dt} = \boldsymbol{\Delta}\cdot \mathrm{grad}\,\mathfrak{V} + \frac{\partial}{\partial t}(\kappa\mathfrak{V}) + \frac{1}{4}\frac{d^3\kappa}{dt^3}r^2 + \frac{d^2\boldsymbol{\delta}}{dt^2}\cdot \boldsymbol{r}\,. \tag{1601}$$

Since $\mathfrak{V}$ satisfies the differential equation (1597), the right-hand side of the foregoing equation vanishes identically, and we conclude that χ_0 is a constant. Hence, our solution for χ is

$$-\tfrac{1}{2}\chi = \kappa\mathfrak{V} + \frac{1}{4}\frac{d^2\kappa}{dt^2}r^2 + \frac{d\boldsymbol{\delta}}{dt}\cdot \boldsymbol{r} + \text{constant}\,. \tag{1602}$$

As we shall see later (§ 71), it is convenient to introduce a variable ϕ defined by

$$\kappa = \phi^2\,. \tag{1603}$$

In terms of ϕ the solution for χ takes the form

$$-\tfrac{1}{2}\chi = \phi^2\mathfrak{V} + \tfrac{1}{2}(\dot\phi^2 + \phi\ddot\phi)r^2 + \dot{\boldsymbol{\delta}}\cdot \boldsymbol{r} + \text{constant}\,. \tag{1604}$$

The solution for σ is now readily obtained, since

$$Q_0 = \kappa(U_0^2 + V_0^2 + W_0^2) = \left|\frac{\boldsymbol{\Delta}}{\phi}\right|^2, \tag{1605}$$

we have (cf. Eq. [1582])

$$Q_0 = \left|\dot\phi\boldsymbol{r} + \frac{1}{\phi}(\boldsymbol{r}\times\boldsymbol{\beta}) + \frac{\boldsymbol{\delta}}{\phi}\right|^2. \tag{1606}$$

Expanding the right-hand side of the foregoing equation, we find that

$$Q_0 = \dot\phi^2 r^2 + \frac{1}{\phi^2}|\boldsymbol{r}\times\boldsymbol{\beta} + \boldsymbol{\delta}|^2 + 2\frac{\dot\phi}{\phi}\boldsymbol{\delta}\cdot\boldsymbol{r}\,. \tag{1607}$$

According to equations (1585), (1604), and (1607), we now have

$$\left.\begin{aligned}\tfrac{1}{2}\sigma = \phi^2\mathfrak{V} + \tfrac{1}{2}\phi\ddot\phi r^2 - \frac{1}{2\phi^2}|\boldsymbol{r}\times\boldsymbol{\beta} + \boldsymbol{\delta}|^2 + \dot{\boldsymbol{\delta}}\cdot\boldsymbol{r}\\ - \frac{\dot\phi}{\phi}\boldsymbol{\delta}\cdot\boldsymbol{r} + \text{constant}\,.\end{aligned}\right\} \tag{1608}$$

Simplifying the last two terms in the foregoing expression, we finally have

$$\left.\begin{aligned}\tfrac{1}{2}\sigma = \phi^2\mathfrak{V} + \tfrac{1}{2}\phi\ddot{\phi}r^2 - \frac{1}{2\phi^2}\,|\boldsymbol{r}\times\boldsymbol{\beta}+\boldsymbol{\delta}|^2 \\ + \phi\frac{d}{dt}\left(\frac{\boldsymbol{\delta}}{\phi}\right)\cdot\boldsymbol{r} + \text{constant}\,.\end{aligned}\right\}\quad(1609)$$

We have now obtained all the general relations that characterize a stellar system described by a spherical distribution of velocities.

71. *Stellar systems with a spherical distribution of the residual velocities and with no translational motions (the case $\boldsymbol{\delta} \equiv 0$).*—We have already seen in § 70 that for stellar systems with a spherical distribution of the residual velocities the motions of the local centroids arise from the superposition of an expansion (or contraction), a rotation, and a translation. The simplest case is, therefore, one for which there exist no translational motions. Accordingly, we shall first examine the case

$$\boldsymbol{\delta} \equiv 0\,. \quad (1610)$$

Then (cf. Eq. [1582])

$$\boldsymbol{\Delta} = \frac{1}{2}\frac{d\kappa}{dt}\boldsymbol{r} + \boldsymbol{r}\times\boldsymbol{\beta}\,, \quad (1611)$$

and our fundamental differential equation is

$$\left(\frac{1}{2}\frac{d\kappa}{dt}\boldsymbol{r} + \boldsymbol{r}\times\boldsymbol{\beta}\right)\cdot\operatorname{grad}\mathfrak{V} + \frac{\partial}{\partial t}(\kappa\mathfrak{V}) + \frac{1}{4}\frac{d^3\kappa}{dt^3}r^2 = 0\,. \quad (1612)$$

Since $\boldsymbol{\beta}$ is a constant vector, we can choose our z-direction to be along the $\boldsymbol{\beta}$-direction. With this choice of the orientation of our co-ordinate system equation (1612) takes the form

$$\left.\begin{aligned}\frac{\partial}{\partial t}(\kappa\mathfrak{V}) + \frac{1}{2}\frac{d\kappa}{dt}\left(x\frac{\partial\mathfrak{V}}{\partial x} + y\frac{\partial\mathfrak{V}}{\partial y} + z\frac{\partial\mathfrak{V}}{\partial z}\right) \\ + \beta\left(y\frac{\partial\mathfrak{V}}{\partial x} - x\frac{\partial\mathfrak{V}}{\partial y}\right) + \frac{1}{4}\frac{d^3\kappa}{dt^3}r^2 = 0\,,\end{aligned}\right\}\quad(1613)$$

where (0, 0, β) has been used to denote $\boldsymbol{\beta}$. Introduce the new variables

$$\tau = \tfrac{1}{2}(x^2 + y^2)\,; \qquad \zeta = \tfrac{1}{2}z^2\,; \qquad \theta = \tan^{-1}\frac{y}{x}\,. \tag{1614}$$

Then

$$\left.\begin{aligned} x\frac{\partial\mathfrak{B}}{\partial x} + y\frac{\partial\mathfrak{B}}{\partial y} &= 2\tau\frac{\partial\mathfrak{B}}{\partial\tau}\,,\\ y\frac{\partial\mathfrak{B}}{\partial x} - x\frac{\partial\mathfrak{B}}{\partial y} &= -\frac{\partial\mathfrak{B}}{\partial\theta}\,,\\ z\frac{\partial\mathfrak{B}}{\partial z} &= 2\zeta\frac{\partial\mathfrak{B}}{\partial\zeta}\,. \end{aligned}\right\} \tag{1615}$$

Equation (1613) now takes the simpler form

$$\frac{\partial}{\partial t}(\kappa\mathfrak{B}) + \frac{d\kappa}{dt}\left(\tau\frac{\partial\mathfrak{B}}{\partial\tau} + \zeta\frac{\partial\mathfrak{B}}{\partial\zeta}\right) - \beta\frac{\partial\mathfrak{B}}{\partial\theta} + \tfrac{1}{2}(\tau+\zeta)\frac{d^3\kappa}{dt^3} = 0\,. \tag{1616}$$

To solve the foregoing linear nonhomogeneous partial differential equation for $\mathfrak{B}$ we proceed as follows:

Differentiating equation (1616) partially with respect to θ, we obtain

$$\frac{\partial^2}{\partial t\partial\theta}(\kappa\mathfrak{B}) + \frac{d\kappa}{dt}\left(\tau\frac{\partial^2\mathfrak{B}}{\partial\tau\partial\theta} + \zeta\frac{\partial^2\mathfrak{B}}{\partial\zeta\partial\theta}\right) - \beta\frac{\partial^2\mathfrak{B}}{\partial\theta^2} = 0\,. \tag{1617}$$

Let

$$\psi = \frac{\partial}{\partial\theta}(\kappa\mathfrak{B})\,. \tag{1618}$$

Equation (1617) now becomes

$$\frac{\partial\psi}{\partial t} + \frac{1}{\kappa}\frac{d\kappa}{dt}\left(\tau\frac{\partial\psi}{\partial\tau} + \zeta\frac{\partial\psi}{\partial\zeta}\right) - \frac{\beta}{\kappa}\frac{\partial\psi}{\partial\theta} = 0\,, \tag{1619}$$

which is a linear homogeneous partial differential equation for ψ. The appropriate subsidiary equations are

$$\frac{dt}{\kappa} = \frac{d\tau}{\tau\dfrac{d\kappa}{dt}} = \frac{d\zeta}{\zeta\dfrac{d\kappa}{dt}} = -\frac{d\theta}{\beta}\,. \tag{1620}$$

Three independent integrals of equation (1620) are readily found. We have

$$\left.\begin{aligned} \frac{\tau}{\kappa} = \text{constant}\,; \qquad \frac{\zeta}{\kappa} &= \text{constant}\,; \\ \theta + \beta\int\frac{dt}{\kappa} &= \text{constant}\,. \end{aligned}\right\} \tag{1621}$$

Hence,

$$\psi = \frac{\partial}{\partial\theta}(\kappa\mathfrak{V}) \equiv F\left(\frac{\tau}{\kappa}\,,\ \frac{\zeta}{\kappa}\,,\ \theta + \beta\int\frac{dt}{\kappa}\right), \tag{1622}$$

where F is an arbitrary function of the arguments specified. Integrating equation (1622), we have

$$\mathfrak{V} = \mathfrak{V}_1(\tau, \zeta, t) + \frac{1}{\kappa}F_1\left(\frac{\tau}{\kappa}\,,\ \frac{\zeta}{\kappa}\,,\ \theta + \beta\int\frac{dt}{\kappa}\right), \tag{1623}$$

where $\mathfrak{V}_1$ is independent of θ and

$$F_1 = \int^{\theta} F d\theta\,. \tag{1624}$$

In equation (1623) we can therefore regard F_1 as an arbitrary function of the arguments specified. Introducing equation (1623) into the original differential equation (1616), we find

$$\frac{\partial}{\partial t}(\kappa\mathfrak{V}_1) + \frac{d\kappa}{dt}\left(\tau\frac{\partial\mathfrak{V}_1}{\partial\tau} + \zeta\frac{\partial\mathfrak{V}_1}{\partial\zeta}\right) + \tfrac{1}{2}(\tau + \zeta)\frac{d^3\kappa_1}{dt^3} = 0\,, \tag{1625}$$

an equation which we have already encountered in Part XI (§§ 59 and 60; see particularly Eq. [1339]). We can therefore immediately write down the general solution of equation (1625). We have (cf. Eqs. [1299] and [1334])

$$\mathfrak{V}_1 = \left(\frac{q_1}{\phi^4} - \frac{\ddot{\phi}}{\phi}\right)\tau + \left(\frac{q_2}{\phi^4} - \frac{\ddot{\phi}}{\phi}\right)\zeta + \frac{1}{\phi^2}F_2\left(\frac{\tau}{\phi^2}\,,\ \frac{\zeta}{\phi^2}\right), \tag{1626}$$

where ϕ^2 is defined as in equation (1603), F_2 is an arbitrary function of the arguments specified, and q_1 and q_2 are arbitrary constants. Combining equations (1623) and (1626), we can clearly write

$$\mathfrak{B} = \left(\frac{q_1}{\phi^4} - \frac{\ddot{\phi}}{\phi}\right)\tau + \left(\frac{q_2}{\phi^4} - \frac{\ddot{\phi}}{\phi}\right)\zeta + \frac{1}{\phi^2}\,\mathfrak{B}^*\left(\frac{\tau}{\phi^2},\ \frac{\zeta}{\phi^2},\ \theta + \beta\int\frac{dt}{\phi^2}\right), \quad (1627)$$

where $\mathfrak{B}^*$ is an arbitrary function of the arguments specified. Equation (1627) represents, therefore, the most general solution of the nonhomogeneous equation (1616).

It is seen that the solution (1627) can be obtained by adding to a particular integral of the nonhomogeneous equation (1616) the general solution of the associated homogeneous equation

$$\frac{\partial}{\partial t}(\kappa\mathfrak{B}) + \frac{d\kappa}{dt}\left(\tau\,\frac{\partial\mathfrak{B}}{\partial\tau} + \zeta\,\frac{\partial\mathfrak{B}}{\partial\zeta}\right) - \beta\,\frac{\partial\mathfrak{B}}{\partial\theta} = 0\,. \qquad (1628)$$

For, since (1628) is a linear homogeneous partial differential equation for $\kappa\mathfrak{B}$ and since, further, the appropriate subsidiary equations are the same as for the equation (1619), it follows that the term involving $\mathfrak{B}^*$ in (1627) is the general solution of the homogeneous equation (1628). Further, the particular integral

$$\left(\frac{q_1}{\phi^4} - \frac{\ddot{\phi}}{\phi}\right)\tau + \left(\frac{q_2}{\phi^4} - \frac{\ddot{\phi}}{\phi}\right)\zeta \qquad (1629)$$

of the equation (1616) can be obtained by seeking a solution of the form

$$C_1(t)\tau + C_2(t)\zeta \qquad (1630)$$

and showing that C_1 and C_2 have exactly the forms required by equation (1629).[73]

Returning to the solution (1627), we see that, since we can absorb the terms $q_1\tau/\phi^4$ and $q_2\zeta/\phi^4$ into $\mathfrak{B}^*/\phi^2$, we can re-write it in the simpler form

$$\mathfrak{B} = -\frac{1}{2}\,\frac{\ddot{\phi}}{\phi}\,r^2 + \frac{1}{\phi^2}\,\mathfrak{B}_1^*\left(\frac{\bar{\omega}}{\phi},\ \frac{z}{\phi},\ \theta + \beta\int\frac{dt}{\phi^2}\right), \qquad (1631)$$

[73] See § 60, particularly p. 522.

where $\mathfrak{B}_1^*$ is an arbirary function of the arguments specified. Finally, according to equations (1604) and (1609), we have

$$-\tfrac{1}{2}\chi = \tfrac{1}{2}\ddot{\phi}\phi r^2 + \mathfrak{B}_1^*\left(\frac{\varpi}{\phi}, \frac{z}{\phi}, \theta + \beta\int\frac{dt}{\phi^2}\right) + \text{constant} \quad (1632)$$

and

$$\tfrac{1}{2}\sigma = -\frac{1}{2}\frac{\beta^2}{\phi^2}\varpi^2 + \mathfrak{B}_1^*\left(\frac{\varpi}{\phi}, \frac{z}{\phi}, \theta + \beta\int\frac{dt}{\phi^2}\right) + \text{constant}\,. \quad (1633)$$

Combining equations (1631) and (1633), we have the relation

$$\tfrac{1}{2}\sigma = \frac{1}{2}\left(\ddot{\phi}\phi - \frac{\beta^2}{\phi^2}\right)\varpi^2 + \tfrac{1}{2}\ddot{\phi}\phi z^2 + \phi^2\mathfrak{B} + \text{constant}\,. \quad (1634)$$

72. *Further discussion of the case* $\boldsymbol{\delta} \equiv 0$: *The spiral phenomenon.*—The solution of the mathematical problem for the case $\boldsymbol{\delta} \equiv 0$ has been obtained in § 71. We shall now examine in some detail the physical characteristics of this solution.

(i) First, let us consider the motions of the local centroids. According to equations (1581) and (1611), we have

$$(U_0, V_0, W_0) = \frac{\boldsymbol{\Delta}}{\phi^2} = \frac{\dot{\phi}}{\phi}\boldsymbol{r} + \frac{1}{\phi^2}(\boldsymbol{r}\times\boldsymbol{\beta})\,. \quad (1635)$$

Remembering that, according to our choice of the orientation of the co-ordinate system, the z-axis is in the $\boldsymbol{\beta}$-direction, we obtain from equation (1635)

$$U_0 = \frac{\dot{\phi}}{\phi}x + \frac{\beta}{\phi^2}y\,; \qquad V_0 = \frac{\dot{\phi}}{\phi}y - \frac{\beta}{\phi^2}x\,; \qquad W_0 = \frac{\dot{\phi}}{\phi}z\,. \quad (1636)$$

If Π_0, Z_0, and Θ_0 denote the components of the motion of local centroid along the radial, the z-, and the transverse directions, respectively, then (as may readily be verified)

$$\Pi_0 = \frac{\dot{\phi}}{\phi}\varpi\,; \qquad Z_0 = \frac{\dot{\phi}}{\phi}z\,; \qquad \Theta_0 = -\frac{\beta}{\phi^2}\varpi\,. \quad (1637)$$

(ii) Consider next the solution for σ. We have (cf. Eq. [1633])

$$\tfrac{1}{2}\sigma = -\frac{1}{2}\frac{\beta^2}{\phi^2}\bar{\omega}^2 + \mathfrak{B}_1^*\left(\frac{\bar{\omega}}{\phi}, \frac{z}{\phi}, \theta + \beta\int\frac{dt}{\phi^2}\right) + \text{constant} ; \qquad (1638)$$

or, since we can absorb the term $-\beta^2\bar{\omega}^2/2\phi^2$ in $\mathfrak{B}_1^*$,

$$\sigma \equiv \sigma\left(\frac{\bar{\omega}}{\phi}, \frac{z}{\phi}, \theta + \beta\int\frac{dt}{\phi^2}\right). \qquad (1639)$$

(iii) The explicit form of the distribution function may be noted. We have

$$\Psi \equiv \Psi(Q + \sigma) , \qquad (1640)$$

where, according to equations (1637) and (1638),

$$\left.\begin{aligned} Q + \sigma = \phi^2\left\{\left(\Pi - \frac{\dot{\phi}}{\phi}\bar{\omega}\right)^2 + \left(Z - \frac{\dot{\phi}}{\phi}z\right)^2 + \left(\Theta + \frac{\beta}{\phi^2}\bar{\omega}\right)^2\right\} \\ + 2\mathfrak{B}_1^* - \frac{\beta^2}{\phi^2}\bar{\omega}^2 . \end{aligned}\right\} \qquad (1641)$$

(iv) If we now specialize our general spherical distribution (Eq. [1579]) to a Maxwellian distribution, then

$$\Psi(Q + \sigma) = e^{-(Q+\sigma)} . \qquad (1642)$$

According to (1641), in our present case we have

$$\Psi = e^{-\phi^2\left\{\left(\Pi - \frac{\dot{\phi}}{\phi}\bar{\omega}\right)^2 + \left(Z - \frac{\dot{\phi}}{\phi}z\right)^2 + \left(\Theta + \frac{\beta}{\phi^2}\bar{\omega}\right)^2\right\} + \frac{\beta^2}{\phi^2}\bar{\omega}^2 - 2\mathfrak{B}_1^*} \qquad (1643)$$

On integration the foregoing equation yields an expression for the number of stars, $\mathfrak{N}$, per unit volume. We find that

$$\mathfrak{N} = \frac{\pi^{3/2}}{\phi^3} e^{(\beta^2/\phi^2)\bar{\omega}^2 - 2\mathfrak{B}_1^*} = \frac{\pi^{3/2}}{\phi^3} e^{-\sigma} . \qquad (1644)$$

According to equation (1639), we can express $\mathfrak{N}$ in the form

$$\mathfrak{N} = \frac{1}{\phi^3}\mathfrak{N}_1\left(\frac{\bar{\omega}}{\phi}, \frac{z}{\phi}, \theta + \beta\int\frac{dt}{\phi^2}\right). \qquad (1645)$$

(v) The physical meaning of ϕ is that its reciprocal is a measure of the dispersion of the residual velocities. For a Maxwellian distribution the mean residual speed in any given direction is given by $\pi^{-1/2}\phi^{-1}$. Thus,

$$\overline{|\Pi - \Pi_0|} = \overline{|Z - Z_0|} = \overline{|\Theta - \Theta_0|} = \frac{1}{\sqrt{\pi}\,\phi} . \qquad (1646)$$

We shall now consider an important characteristic of the function σ.[74] According to equation (1639), σ involves the time only *implicitly* through the function ϕ. The nature of the dependence of σ on t is best understood by considering the locus of points at which σ takes some preassigned value as the time varies. Let us first consider this locus in the (x, y) plane (i.e., $z = Z_0 = 0$). Since σ now depends only on

$$\frac{\bar{\omega}}{\phi} \quad \text{and} \quad \theta + \beta \int \frac{dt}{\phi^2} , \qquad (1647)$$

it follows that for prescribed values for these two quantities σ will always take the same constant value.[75] Thus, let $\sigma = \sigma_1$ when

$$\frac{\bar{\omega}}{\phi} = c_1 ; \qquad \theta + \beta \int \frac{dt}{\phi^2} = c_2 . \qquad (1648)$$

The *trajectory* described by the point at which σ takes this prescribed value σ_1 is, therefore, defined parametrically by the relations

$$\bar{\omega} = c_1 \phi(t) ; \qquad \theta = c_2 - \beta \int \frac{dt}{\phi^2} ; \qquad (1649)$$

in other words, the required trajectory is obtained by eliminating t between the relations (1649), keeping c_1 and c_2 fixed. We shall now show that the locus (1649) is a *general spiral:*

[74] The remarks which follow apply equally well also for the function $\mathfrak{N}_1$ (Eq. [1645]).

[75] We are assuming that σ is a one-valued continuous function of the arguments.

According to equations (1649), the increments $d\bar{\omega}$ and $d\theta$ in $\bar{\omega}$ and θ which result from an increment dt in t are

$$d\bar{\omega} = c_1\dot{\phi}dt \; ; \qquad d\theta = -\beta\frac{dt}{\phi^2} . \tag{1650}$$

Let us assume that the rotation is in the positive counterclockwise direction. Then, according to (1637), β is negative. It now follows from (1650) that

$$\left.\begin{aligned} d\bar{\omega} &> 0 \,, & d\theta &> 0 \,, & \dot{\phi} &> 0 \,, & \beta &< 0 \,, \\ d\bar{\omega} &< 0 \,, & d\theta &> 0 \,, & \dot{\phi} &< 0 \,, & \beta &< 0 \,. \end{aligned}\right\} \tag{1651}$$

Hence, *the locus (1649) is a general spiral described outward in the direction of rotation if $\dot{\phi}$ is positive and described inward if $\dot{\phi}$ is negative.* (See also Figs. 9*a* and 9*b*.)

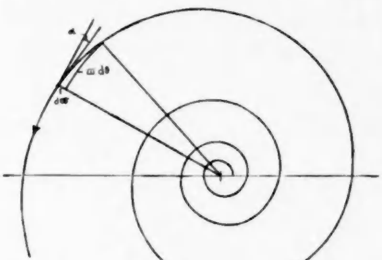

FIG. 9*a*

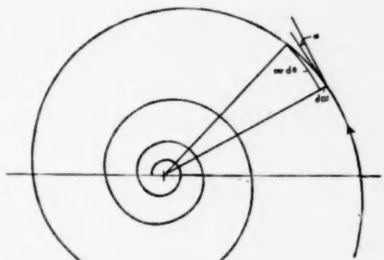

FIG. 9*b*

We may first notice how the two cases $\dot{\phi} > 0$ and $\dot{\phi} < 0$ are physically distinguished. If $\dot{\phi} > 0$, Π_0 and Z_0 are both positive and correspond to a general *expansion* of the system; further, according to (1646), the mean residual speed decreases with time. If $\dot{\phi} < 0$, then the converse is true.

Equations (1650) enable us to derive one further result of considerable importance in the interpretation of our present solution (§ 75). Using the first of the equations (1649), we can re-write equations (1650) as

$$d\bar{\omega} = \bar{\omega}\frac{\dot{\phi}}{\phi}dt \; ; \qquad \bar{\omega}d\theta = -\frac{\beta}{\phi^2}\bar{\omega}dt \; ; \tag{1652}$$

or, according to equations (1637), we have

$$d\bar{\omega} = \Pi_0 dt\,; \qquad \bar{\omega} d\theta = \Theta_0 dt\,. \tag{1653}$$

In other words, *the motion of points of constant σ are governed directly by the motions of the appropriate local centroids.*

The angle of the spiral (1649) is readily found. Let us denote the angle by α. By definition (see Fig. 9)

$$\tan\alpha = \frac{d\bar{\omega}}{\bar{\omega} d\theta}\,; \tag{1654}$$

or, according to (1653) and (1637),

$$\tan\alpha = \frac{\Pi_0}{\Theta_0} = -\frac{\dot{\phi}\phi}{\beta} = -\frac{1}{2\beta}\,\frac{d\phi^2}{dt}\,. \tag{1655}$$

The equations of the trajectories can be explicitly given if ϕ is known as a function of time; but within the framework of our present theory ϕ is left undetermined and can be any arbitrary function of time. However, a case of special interest arises when ϕ^2 varies linearly with the time. Then

$$\phi^2 = A(t - t_0)\,, \tag{1656}$$

where A and t_0 are constants. According to equation (1649), we now have

$$\bar{\omega} = c_1\sqrt{A(t - t_0)}\,; \qquad \theta = c_2 - \frac{\beta}{A}\log(t - t_0)\,. \tag{1657}$$

The elimination of t between the equations (1657) is readily effected. We find

$$\bar{\omega} = c_1\sqrt{A}\, e^{-A(\theta - c_2)/2\beta}\,. \tag{1658}$$

Further, when ϕ^2 has the form (1656) (cf. Eq. [1655]),

$$\frac{\Pi_0}{\Theta_0} = -\frac{A}{2\beta} = \text{constant}\,. \tag{1659}$$

Hence, we can re-write (1658) in the form

$$\bar{\omega} = \bar{\omega}_0 e^{(\Pi_0/\Theta_0)\theta} \, , \tag{1660}$$

which is the equation of a *logarithmic spiral*.[76]

So far, we have considered the trajectories of points of constant σ only in the (x, y) plane. The general case, however, presents no essentially new features. If $\sigma = \sigma_1$ when

$$\frac{\bar{\omega}}{\phi} = c_1 \, ; \qquad \frac{z}{\phi} = c_3 \, ; \qquad \theta + \beta \int \frac{dt}{\phi^2} = c_2 \, , \tag{1661}$$

then the locus of points at which $\sigma = \sigma_1$ is obtained by eliminating t between the relations (1661), keeping c_1, c_2, and c_3 fixed. If $d\bar{\omega}$, dz, and $d\theta$ are the increments in $\bar{\omega}$, z, and θ, owing to an increment dt in t, then

$$d\bar{\omega} = c_1\dot{\phi}dt \, ; \qquad dz = c_3\dot{\phi}dt \, ; \qquad d\theta = -\beta \frac{dt}{\phi_2} \, , \tag{1662}$$

Hence,

$$\frac{d\bar{\omega}}{dz} = \frac{c_1}{c_3} = \frac{\bar{\omega}}{z} = \text{constant} \, . \tag{1663}$$

Thus, the point moves radially[77] from the center, while the projection of the trajectory in the (x, y) plane is a general spiral.

The bearing of the spiral phenomenon which we have now encountered on the interpretation of the spiral nebulae is considered in § 75.

73. *More general spiral systems.*—In §§ 71 and 72 we restricted ourselves to the case $\boldsymbol{\delta} \equiv 0$. The discussion of this case disclosed an essentially new feature of our theory, namely, the occurrence of a spiral phenomenon within the framework of our fundamental kinematical postulates. We shall now show that the spiral phenomenon encountered in §§ 71 and 72 is only a very special example of a much more general characteristic of stellar systems described by a spherical distribution of the residual velocities.

[76] This is what we should have expected, since, according to Eqs. (1655) and (1656), the angle α is a constant.

[77] Outward if $\dot{\phi} > 0$, and inward if $\dot{\phi} < 0$.

As we have already seen (§ 70), the solution of the general problem hinges on the differential equation (cf. Eq. [1597])

$$\left.\begin{aligned}\left(\frac{1}{2}\frac{d\kappa}{dt}\boldsymbol{r}+\boldsymbol{r}\times\boldsymbol{\beta}+\boldsymbol{\delta}\right)\cdot\operatorname{grad}\mathfrak{B}+\frac{\partial}{\partial t}(\kappa\mathfrak{B})&\\+\frac{1}{4}\frac{d^3\kappa}{dt^3}r^2+\frac{d^2\boldsymbol{\delta}}{dt^2}\cdot\boldsymbol{r}=0\,.&\end{aligned}\right\}\qquad(1664)$$

Since $\boldsymbol{\beta}$ is a constant vector (Eq. [1588]), we can choose the z-axis of our co-ordinate system to lie along the $\boldsymbol{\beta}$-direction. With this choice of the orientation, the vector $\boldsymbol{\beta}$ has the form $(0, 0, \beta)$, and equation (1664) reduces to

$$\left.\begin{aligned}\left(\frac{1}{2}\frac{d\kappa}{dt}x+\beta y+\delta_1\right)\frac{\partial\mathfrak{B}}{\partial x}+\left(\frac{1}{2}\frac{d\kappa}{dt}y-\beta x+\delta_2\right)\frac{\partial\mathfrak{B}}{\partial y}&\\+\left(\frac{1}{2}\frac{d\kappa}{dt}z+\delta_3\right)\frac{\partial\mathfrak{B}}{\partial z}+\frac{\partial}{\partial t}(\kappa\mathfrak{B})+\tfrac{1}{4}(x^2+y^2+z^2)\frac{d^3\kappa}{dt^3}&\\+x\ddot{\delta}_1+y\ddot{\delta}_2+z\ddot{\delta}_3=0\,.&\end{aligned}\right\}\qquad(1665)$$

Now, equation (1665) is a nonhomogeneous partial differential equation for $\mathfrak{B}$. To solve this equation, we shall first obtain the general solution of the associated homogeneous equation and then seek a particular integral of the nonhomogeneous equation.

I. *The solution of the homogeneous equation.*—The homogeneous equation associated with equation (1665) can be written as

$$\left.\begin{aligned}\left(\frac{1}{2}\frac{d\kappa}{dt}x+\beta y+\delta_1\right)\frac{\partial}{\partial x}(\kappa\mathfrak{B})+\left(\frac{1}{2}\frac{d\kappa}{dt}y-\beta x+\delta_2\right)\frac{\partial}{\partial y}(\kappa\mathfrak{B})&\\+\left(\frac{1}{2}\frac{d\kappa}{dt}z+\delta_3\right)\frac{\partial}{\partial z}(\kappa\mathfrak{B})+\kappa\frac{\partial}{\partial t}(\kappa\mathfrak{B})=0\,,&\end{aligned}\right\}\qquad(1666)$$

where it will be recalled that δ_1, δ_2, and δ_3 are functions of time. Equation (1666) is seen to be a linear homogeneous partial differential equation for $\kappa\mathfrak{B}$. To solve this equation, we first write down the appropriate subsidiary equations, which are

$$\frac{dx}{\frac{1}{2}\frac{d\kappa}{dt}x+\beta y+\delta_1}=\frac{dy}{\frac{1}{2}\frac{d\kappa}{dt}y-\beta x+\delta_2}=\frac{dz}{\frac{1}{2}\frac{d\kappa}{dt}z+\delta_3}=\frac{dt}{\kappa}\,.\qquad(1667)$$

These equations can be expressed alternatively as

$$\left.\begin{aligned}\kappa\frac{dx}{dt} &= \frac{1}{2}\frac{d\kappa}{dt}x+\beta y+\delta_1\,,\\ \kappa\frac{dy}{dt} &= \frac{1}{2}\frac{d\kappa}{dt}y-\beta x+\delta_2\,,\\ \kappa\frac{dz}{dt} &= \frac{1}{2}\frac{d\kappa}{dt}z+\delta_3\,.\end{aligned}\right\}\tag{1668}$$

To obtain three independent integrals of the foregoing equations we proceed as follows:

First, we notice that equations (1668) can be re-written more simply in the forms

$$\left.\begin{aligned}\kappa^{3/2}\frac{d}{dt}\left(\frac{x}{\sqrt{\kappa}}\right) &= \beta y+\delta_1\,,\\ \kappa^{3/2}\frac{d}{dt}\left(\frac{y}{\sqrt{\kappa}}\right) &= -\beta x+\delta_2\,,\\ \kappa^{3/2}\frac{d}{dt}\left(\frac{z}{\sqrt{\kappa}}\right) &= \delta_3\,.\end{aligned}\right\}\tag{1669}$$

The following change of variables now suggests itself:

$$\xi=\frac{x}{\sqrt{\kappa}}\,;\qquad \eta=\frac{y}{\sqrt{\kappa}}\,;\qquad \zeta=\frac{z}{\sqrt{\kappa}}\,.\tag{1670}$$

In terms of these new variables equations (1669) reduce to

$$\left.\begin{aligned}\kappa\frac{d\xi}{dt} &= \beta\eta+\frac{\delta_1}{\sqrt{\kappa}}\,,\\ \kappa\frac{d\eta}{dt} &= -\beta\xi+\frac{\delta_2}{\sqrt{\kappa}}\,,\\ \kappa\frac{d\zeta}{dt} &= \frac{\delta_3}{\sqrt{\kappa}}\,.\end{aligned}\right\}\tag{1671}$$

We shall now introduce a new independent variable ι, defined according to the relations

$$d\iota = \frac{dt}{\kappa}; \qquad \iota = \int_{t_0}^{t} \frac{dt}{\kappa}, \tag{1672}$$

where t_0 corresponds to some appropriately chosen origin of time. Equations (1671) now take the simplified forms

$$\frac{d\xi}{d\iota} = \beta\eta + \frac{\delta_1}{\sqrt{\kappa}}; \qquad \frac{d\eta}{d\iota} = -\beta\xi + \frac{\delta_2}{\sqrt{\kappa}}, \tag{1673}$$

and

$$\frac{d\zeta}{d\iota} = \frac{\delta_3}{\sqrt{\kappa}}. \tag{1674}$$

Equations (1673) can be solved by the method of the variation of parameters: The solution of the associated homogeneous system

$$\frac{d\xi}{d\iota} = \beta\eta; \qquad \frac{d\eta}{d\iota} = -\beta\xi \tag{1675}$$

is

$$\left.\begin{aligned} \xi &= A\cos\beta\iota + B\sin\beta\iota, \\ \eta &= -A\sin\beta\iota + B\cos\beta\iota, \end{aligned}\right\} \tag{1676}$$

where A and B are constants. For solving the nonhomogeneous system (1673), we assume that the solution is of the form (1676), where A and B are now no longer regarded as constants but as functions of ι. Introducing equations (1676) in equations (1673), we obtain

$$\left.\begin{aligned} \frac{dA}{d\iota}\cos\beta\iota + \frac{dB}{d\iota}\sin\beta\iota &= \frac{\delta_1}{\sqrt{\kappa}}, \\ -\frac{dA}{d\iota}\sin\beta\iota + \frac{dB}{d\iota}\cos\beta\iota &= \frac{\delta_2}{\sqrt{\kappa}}; \end{aligned}\right\} \tag{1677}$$

or, solving for $dA/d\iota$ and $dB/d\iota$, we have

$$\left.\begin{aligned}\frac{dA}{d\iota} &= \frac{1}{\sqrt{\kappa}}(\delta_1 \cos \beta\iota - \delta_2 \sin \beta\iota)\,,\\ \frac{dB}{d\iota} &= \frac{1}{\sqrt{\kappa}}(\delta_1 \sin \beta\iota + \delta_2 \cos \beta\iota)\,.\end{aligned}\right\} \tag{1678}$$

Integrating the foregoing equations, we have

$$A = A_0 + J_1\,; \qquad B = B_0 + J_2\,, \tag{1679}$$

where A_0 and B_0 are arbitrary constants and

$$\left.\begin{aligned}J_1 &= \int \frac{1}{\sqrt{\kappa}}(\delta_1 \cos \beta\iota - \delta_2 \sin \beta\iota)d\iota\,,\\ J_2 &= \int \frac{1}{\sqrt{\kappa}}(\delta_1 \sin \beta\iota + \delta_2 \cos \beta\iota)d\iota\,.\end{aligned}\right\} \tag{1680}$$

In equations (1680) the integrals on the right-hand sides are indefinite integrals. Combining equations (1676) and (1679), we obtain the required solutions for ξ and η. We have

$$\left.\begin{aligned}\xi - \xi_0 &= A_0 \cos \beta\iota + B_0 \sin \beta\iota\,,\\ \eta - \eta_0 &= -A_0 \sin \beta\iota + B_0 \cos \beta\iota\,,\end{aligned}\right\} \tag{1681}$$

where

$$\left.\begin{aligned}\xi_0 &= J_1 \cos \beta\iota + J_2 \sin \beta\iota\,,\\ \eta_0 &= -J_1 \sin \beta\iota + J_2 \cos \beta\iota\,.\end{aligned}\right\} \tag{1682}$$

From equations (1681) we readily derive two first-integrals of the subsidiary equations (1667). According to (1681), we clearly have

$$(\xi - \xi_0)^2 + (\eta - \eta_0)^2 = A_0^2 + B_0^2 = \text{constant}\,, \tag{1683}$$

which is one first-integral. Again, according to equations (1681), we have

$$\frac{\eta - \eta_0}{\xi - \xi_0} = \frac{-A_0 \sin \beta\iota + B_0 \cos \beta\iota}{A_0 \cos \beta\iota + B_0 \sin \beta\iota}\,, \tag{1684}$$

or, alternatively, as

$$\frac{\eta - \eta_0}{\xi - \xi_0} = \frac{\tan \Sigma - \tan \beta\iota}{1 + \tan \Sigma \tan \beta\iota} = \tan (\Sigma - \beta\iota) , \tag{1685}$$

where

$$\tan \Sigma = \frac{B_0}{A_0} = \text{constant} . \tag{1686}$$

From equation (1685) it now readily follows that

$$\vartheta + \beta\iota = \text{constant} , \tag{1687}$$

where

$$\vartheta = \tan^{-1} \frac{\eta - \eta_0}{\xi - \xi_0} . \tag{1688}$$

Equation (1687) is another first-integral of the subsidiary equations.

Consider next equation (1674). This equation admits of immediate integration, and we can write

$$\zeta - \zeta_0 = \text{constant} , \tag{1689}$$

where

$$\zeta_0 = \int \frac{\delta_3}{\sqrt{\kappa}} \, d\iota . \tag{1690}$$

In equation (1690) the integral on the right-hand side is an indefinite integral.

We have now obtained three independent first-integrals of the subsidiary equations, and consequently we can immediately write down the general solution of the homogeneous equation (1666). We have

$$\mathfrak{B} = \frac{1}{\kappa} \mathfrak{B}^* \{ (\xi - \xi_0)^2 + (\eta - \eta_0)^2 ; \ \vartheta + \beta\iota ; \ \zeta - \zeta_0 \} , \tag{1691}$$

where $\mathfrak{B}^*$ is an arbitrary function of the arguments specified.

We first notice that when $\boldsymbol{\delta} \equiv 0$,

$$\xi_0 = \eta_0 = \zeta_0 \equiv 0 ; \qquad \vartheta \equiv \theta \qquad (\boldsymbol{\delta} \equiv 0) , \tag{1692}$$

and the solution (1691) reduces to

$$\mathfrak{B} = \frac{1}{\kappa}\,\mathfrak{B}^*(\xi^2 + \eta^2\,;\ \theta + \beta\iota\,;\ \zeta)\,; \tag{1693}$$

this is in agreement with the results of § 71 (cf. Eq. [1627]), as it should be.

Returning to the general solution (1691), consider the function $\mathfrak{B}^*$ in the (ξ, η, ζ) space. The character of this function is best studied by considering the locus of points at which $\mathfrak{B}^*$ takes some specified constant value. Such loci will be defined parametrically by relations of the form

$$(\xi - \xi_0)^2 + (\eta - \eta_0)^2 = c_1^2\,; \qquad \vartheta + \beta\iota = c_2\,, \tag{1694}$$

and

$$\zeta - \zeta_0 = c_3\,, \tag{1695}$$

where c_1, c_2, and c_3 are constants.

Consider, first, the projection of the locus in the (ξ, η) plane. This is a curve defined parametrically by the relations (1694). This locus, (1694), has a simple geometrical interpretation (see Fig. 10):

According to equations (1680) and (1682), the point (ξ_0, η_0) will describe some locus (as ι varies) in the (ξ, η) plane. The explicit form which this locus will take will depend on δ_1 and δ_2; but when δ_1 and δ_2 are known, the (ξ_0, η_0) locus will be uniquely determined (see below). We can regard this (ξ_0, η_0) locus as being described by a moving representative point, P_0. The locus (1694) can be derived now from the (ξ_0, η_0) locus by attaching to the representative point P_0 an "arm" P_0P, of length c_1 and allowing it to rotate about P_0 with a constant angular velocity $-\beta$. The curve described by P in this manner is the required locus.[78] (In §§ 74 and 75 we consider these loci more fully.)

From our discussion in the preceding paragraph it follows that the

[78] It is seen that the curves described in this manner give rise to a new class of transcendental curves. These belong, however, to the same family of curves as cycloids, epicycloids, cardioids, etc. An important contribution to the theory of rolling curves, which is of interest in our present connection, is that due to Clerk Maxwell (see his *Collected Papers*, **1**, 4–29, 1890, Cambridge England).

locus of points in the (ξ, η) plane at which $\mathfrak{V}^*$ takes some prescribed constant value is a transcendental curve of the same general nature as an epitrochoid or a hypotrochoid. To transform these loci in the (ξ, η) plane to loci in the (x, y) plane we should apply the transformation (cf. Eq. [1670])

$$x = \sqrt{\kappa}\,\xi\,; \qquad y = \sqrt{\kappa}\,\eta\,, \tag{1696}$$

where κ is itself a function of ι (see Eq. [1672]). The result of the transformation (1696) will be that the loci in the (x, y) plane will be

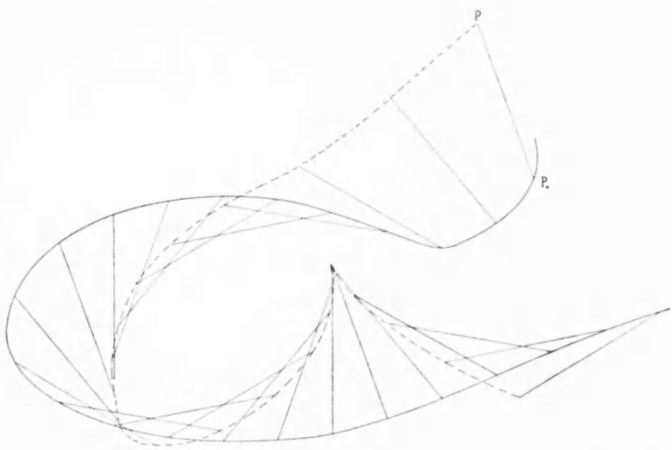

FIG. 10.—The geometrical construction for deriving the trajectories of points of constant σ in the (ξ, η) plane from a basic (ξ_0, η_0) locus. While P_0 describes the (ξ_0, η_0) locus (the full line curve) the "arm" $P_0\,P$ rotates about P_0 at a constant rate and traces a possible trajectory (the dotted curve).

spirals which may have kinks or may even interpenetrate themselves.[79] We shall postpone to § 75 the further consideration of these and related matters.

II. *A particular integral of the nonhomogeneous equation.*—In the two previous cases (§§ 59, 60, and 71) where we have succeeded in solving for the most general solutions of nonhomogeneous partial differential equations, we have seen that the general solution con-

[79] This becomes evident when we realize that the "smooth" spirals considered in § 72 are derived from *circles* in the (ξ, η) plane.

sists of the solution of the homogeneous equation and a particular integral of the form

$$\tfrac{1}{2}C_1(t)(x^2 + y^2) + \tfrac{1}{2}C_2(t)z^2 , \tag{1697}$$

where C_1 and C_2 are functions of time only. Hence, in order to complete our solution of the nonhomogeneous equation (1665), we shall seek a particular integral of this equation, again of the form (1697). We realize, of course, that we may not obtain in this manner the most general particular integral (cf. § 60). (The most general particular integral is obtained in § 76.)

Introducing (1697) into the differential equation (1665), we find

$$\left.\begin{aligned} &\tfrac{1}{2}(C_1x^2 + C_1y^2 + C_2z^2)\frac{d\kappa}{dt} + (C_1\delta_1x + C_1\delta_2y + C_2\delta_3z) \\ &\quad + \frac{1}{2}\frac{\partial}{\partial t}(\kappa[C_1x^2 + C_1y^2 + C_2z^2]) + \tfrac{1}{4}(x^2 + y^2 + z^2)\frac{d^3\kappa}{dt^3} \\ &\qquad + (x\ddot{\delta}_1 + y\ddot{\delta}_2 + z\ddot{\delta}_3) = 0 . \end{aligned}\right\} \tag{1698}$$

Equating the coefficients of x^2, y^2, z^2, x, y, and z, we obtain

$$\left.\begin{aligned} 2\frac{d\kappa}{dt}C_1 + \kappa\frac{dC_1}{dt} + \frac{1}{2}\frac{d^3\kappa}{dt^3} = 0 , \\ 2\frac{d\kappa}{dt}C_2 + \kappa\frac{dC_2}{dt} + \frac{1}{2}\frac{d^3\kappa}{dt^3} = 0 , \end{aligned}\right\} \tag{1699}$$

and

$$C_1\delta_1 + \ddot{\delta}_1 = 0 ; \qquad C_1\delta_2 + \ddot{\delta}_2 = 0 ; \qquad C_2\delta_3 + \ddot{\delta}_3 = 0 . \tag{1700}$$

From equations (1699) it follows that

$$C_1 = \pm\frac{q_1^2}{\phi^4} - \frac{\ddot{\phi}}{\phi} ; \qquad C_2 = \pm\frac{q_2^2}{\phi^4} - \frac{\ddot{\phi}}{\phi} , \tag{1701}$$

where

$$\phi^2 = \kappa \tag{1702}$$

and q_1 and q_2 are two arbitrary *real* numbers.

Combining equations (1700) and (1701), we can express the δ's

in terms of ϕ, using the results of § 63. Thus, considering the equations for δ_1, we have

$$-\frac{\ddot{\delta}_1}{\delta_1} = \pm\frac{q_1^2}{\phi^4} - \frac{\ddot{\phi}}{\phi}. \tag{1703}$$

According to equation (1380), we now have

$$\int^t \frac{dt}{\phi^2} = \pm\frac{1}{q_1}\sin^{-1}\left(\frac{q_1}{q_0}\frac{\delta_1}{\phi}\right) \tag{1704}$$

or

$$\int^t \frac{dt}{\phi^2} = \pm\frac{1}{q_1}\sinh^{-1}\left(\frac{q_1}{q_0}\frac{\delta_1}{\phi}\right) \quad \text{or} \quad \pm\frac{1}{q_1}\cosh^{-1}\left(\frac{q_1}{q_0}\frac{\delta_1}{\phi}\right), \tag{1705}$$

depending on whether we have the plus or the minus sign in equation (1703). In equations (1704) and (1705) q_0 is an arbitrary real constant. We can express the foregoing equations alternatively in the form

$$\delta_1 = \delta_{10}\phi \begin{Bmatrix} \sin \\ \sinh \\ \cosh \end{Bmatrix} q_1\int^t \frac{dt}{\phi^2}, \tag{1706}$$

where δ_{10} is an arbitrary constant. Introducing, now, the variable ι (Eq. [1672]), we can re-write (1706) more conveniently as

$$\delta_1 = \delta_{10}\phi \begin{Bmatrix} \sin \\ \sinh \\ \cosh \end{Bmatrix} (q_1\iota + \iota_1), \tag{1707}$$

where ι_1 is an arbitrary constant. Similarly, from the equations for δ_2 and δ_3 we obtain

$$\delta_2 = \delta_{20}\phi \begin{Bmatrix} \sin \\ \sinh \\ \cosh \end{Bmatrix} (q_1\iota + \iota_2) \tag{1708}$$

and

$$\delta_3 = \delta_{30}\phi \begin{Bmatrix} \sin \\ \sinh \\ \cosh \end{Bmatrix} (q_2\iota + \iota_3), \tag{1709}$$

where δ_{20}, δ_{30}, ι_2, and ι_3 are all constants of integration.

We now see that the assumption of the existence of a particular

integral of the form (1697) restricts the functions δ_1, δ_2, and δ_3. Consequently, it appears that the particular integral

$$\frac{1}{2}\left(\pm\frac{q_1^2}{\phi^4}-\frac{\ddot{\phi}}{\phi}\right)(x^2+y^2)+\frac{1}{2}\left(\pm\frac{q_2^2}{\phi^4}-\frac{\ddot{\phi}}{\phi}\right)z^2 \tag{1710}$$

is not the most general particular integral of the equation (1665).[80] However, since (1710) is identical in form with the most general particular integral for the case $\boldsymbol{\delta} \equiv 0$, we shall use the particular integral (1710) and its consequences (Eqs. [1707]–[1709]) in our further discussions.

74. *Further discussion of the solution of the nonhomogeneous equation (1665).*—We shall first summarize the results of our discussion so far:

(i) The *complementary function*[81] can be expressed in the form (cf. Eqs. [1670], [1691], and [1702])

$$\mathfrak{V}_c = \frac{1}{\phi^2}\mathfrak{V}^*\{(\xi-\xi_0)^2+(\eta-\eta_0)^2\,;\ \vartheta+\beta\iota\,;\ \zeta-\zeta_0\}\,, \tag{1711}$$

where

$$x=\xi\phi\,;\qquad y=\eta\phi\,;\qquad z=\zeta\phi\,. \tag{1712}$$

Further, according to equations (1680), (1682), and (1688)–(1690), we have

$$\left.\begin{aligned}\xi_0 = &\left\{\int\left(\frac{\delta_1}{\phi}\cos\beta\iota-\frac{\delta_2}{\phi}\sin\beta\iota\right)d\iota\right\}\cos\beta\iota\\ &\qquad+\left\{\int\left(\frac{\delta_1}{\phi}\sin\beta\iota+\frac{\delta_2}{\phi}\cos\beta\iota\right)d\iota\right\}\sin\beta\iota\,,\end{aligned}\right\} \tag{1713}$$

$$\left.\begin{aligned}\eta_0 = -&\left\{\int\left(\frac{\delta_1}{\phi}\cos\beta\iota-\frac{\delta_2}{\phi}\sin\beta\iota\right)d\iota\right\}\sin\beta\iota\\ &\qquad+\left\{\int\left(\frac{\delta_1}{\phi}\sin\beta\iota+\frac{\delta_2}{\phi}\cos\beta\iota\right)d\iota\right\}\cos\beta\iota\,,\end{aligned}\right\} \tag{1714}$$

$$\zeta_0 = \int\frac{\delta_3}{\phi}\,d\iota\,, \tag{1715}$$

[80] We now appreciate the relevance of the comments made in § 60 (see n. 52). For the most general particular integral of equation (1665) see § 76.

[81] The complementary function is defined as the most general solution of the associated homogeneous equation.

and

$$\vartheta = \tan^{-1} \frac{\eta - \eta_0}{\xi - \xi_0} . \tag{1716}$$

Finally, the variable ι is related to t according to the relations (cf. Eqs. [1672] and [1702])

$$dt = \phi^2 d\iota ; \qquad \iota = \int_0^t \frac{dt}{\phi^2} . \tag{1717}$$

(ii) Equation (1665) admits of a *particular integral*

$$\mathfrak{B}_p = \frac{1}{2}\left(\pm \frac{q_1^2}{\phi^4} - \frac{\ddot{\phi}}{\phi}\right)(x^2 + y^2) + \frac{1}{2}\left(\pm \frac{q_2^2}{\phi^4} - \frac{\ddot{\phi}}{\phi}\right)z^2 , \tag{1718}$$

where q_1 and q_2 are arbitrary real numbers.

(iii) The existence of the integral (1718) implies that δ_1, δ_2, and δ_3 have the forms (cf. Eqs. [1707]–[1709])

$$\left.\begin{aligned}
\frac{\delta_1}{\phi} &= \delta_{10} \begin{Bmatrix} \sin \\ \sinh \\ \cosh \end{Bmatrix} (q_1\iota + \iota_1) , \\
\frac{\delta_2}{\phi} &= \delta_{20} \begin{Bmatrix} \sin \\ \sinh \\ \cosh \end{Bmatrix} (q_1\iota + \iota_2) , \\
\frac{\delta_3}{\phi} &= \delta_{30} \begin{Bmatrix} \sin \\ \sinh \\ \cosh \end{Bmatrix} (q_2\iota + \iota_3) ,
\end{aligned}\right\} \tag{1719}$$

where δ_{10}, δ_{20}, δ_{30}, ι_1, ι_2, and ι_3 are constants of integration. The question as to whether we should use the circular or the hyperbolic functions in (1719) depends on the sign with which the terms q_1^2/ϕ^4 and q_2^2/ϕ^4 occur in (1718). Thus, if q_1^2/ϕ^4 occurs with a positive sign in (1718), then the circular functions should be used in the expressions for δ_1/ϕ and δ_2/ϕ; otherwise, the hyperbolic functions should be used. A similar rule applies also for δ_3/ϕ.

(iv) The restrictions on the δ's required by equation (1719) arise

from the assumption of the existence of a particular integral of the form

$$\tfrac{1}{2}C_1(t)(x^2 + y^2) + \tfrac{1}{2}C_2(t)z^2 \tag{1720}$$

and hence do not correspond to the most general particular integral of equation (1665). If, however, we restrict ourselves to the case (1718),[82] the solution of nonhomogeneous equation (1665) can be written as

$$\mathfrak{B} = \mathfrak{B}_c + \mathfrak{B}_p\,. \tag{1721}$$

As we have already indicated, we shall base our further discussion on the solution (1721) for $\mathfrak{B}$:

I. *The* (ξ_0, η_0) *locus*.—From our discussion in § 73 (pp. 595–596) it follows that the most important features of the solution are derivable from the (ξ_0, η_0) locus in the (ξ, η) plane. This locus, defined parametrically by the relations (1713) and (1714), depends on δ_1/ϕ and δ_2/ϕ. Since, however, we now know δ_1/ϕ and δ_2/ϕ explicitly as functions of ι (cf. Eq. [1719]), the required integrations in (1713) and (1714) can be effected and the nature of the locus (ξ_0, η_0) determined.

We first notice that, according to (1719), we have to distinguish between the two cases

$$\frac{\delta_1}{\phi} = \delta_{10} \sin\,(q_1\iota + \iota_1)\;; \qquad \frac{\delta_2}{\phi} = \delta_{20} \sin\,(q_1\iota + \iota_2) \tag{1722}$$

and

$$\frac{\delta_1}{\phi} = \delta_{10} \left\{ \begin{matrix} \sinh \\ \cosh \end{matrix} \right\} (q_1\iota + \iota_1)\;; \qquad \frac{\delta_2}{\phi} = \delta_{20} \left\{ \begin{matrix} \sinh \\ \cosh \end{matrix} \right\} (q_1\iota + \iota_2)\,. \tag{1723}$$

In either case, we can eliminate ι between the equations for δ_1/ϕ and δ_2/ϕ and obtain a δ-*locus* in the (ξ, η) plane.[83] If equation (1722)

[82] The consequences of this restriction are further discussed in § 76 where the most general particular integral is obtained.

[83] According to the convention we have adopted, $\boldsymbol{\delta}$ is a vector with components δ_1, δ_2, and δ_3 along the three principal axes.

is valid, the δ-locus is readily seen to be an ellipse. On the other hand, if (1723) is valid, then, the δ-locus is a hyperbola.[84] We shall therefore consider these two cases separately:

Case (i): The δ-locus an ellipse, and $|q_1| \neq |\beta|$.—Substituting for δ_1/ϕ and δ_2/ϕ according to (1722) in equations (1713) and (1714) and carrying out the necessary integrations, we find that

$$\left.\begin{aligned} \xi_0 &= \frac{1}{\beta^2 - q_1^2}\left[\delta_{10} q_1 \cos(q_1 t + t_1) + \delta_{20}\beta \sin(q_1 t + t_2)\right] , \\ \eta_0 &= \frac{1}{\beta^2 - q_1^2}\left[-\delta_{10}\beta \sin(q_1 t + t_1) + \delta_{20} q_1 \cos(q_1 t + t_2)\right] . \end{aligned}\right\} \quad (1724)$$

Hence, *the (ξ_0, η_0) locus is, in general, an ellipse.* We shall therefore refer to this locus as the (ξ_0, η_0) *ellipse.*

Now, we have already seen that the δ-locus, defined parametrically by the relations (1722), is also an ellipse. Since δ_{10}, δ_{20}, t_1, and t_2 are constants of integration, the principal axes of the δ-*ellipse* are along fixed directions. Consequently, we can apply a rotation to the co-ordinate system about the z-direction such that the ξ- and the η-directions are along the principal axes of the δ-ellipse. If the co-ordinate axes are oriented in this manner, the δ-locus will be defined by the relations

$$\frac{\delta_1}{\phi} = \delta_{10} \cos(q_1 t + t_0) \; ; \qquad \frac{\delta_2}{\phi} = \delta_{20} \sin(q_1 t + t_0) \; . \qquad (1725)$$

The corresponding expressions for ξ_0 and η_0 can be readily derived from (1724) by putting

$$t_1 = t_0 + \frac{\pi}{2} \; ; \qquad t_2 = t_0 \; . \qquad (1726)$$

[84] We notice here a certain analogy between the equations

$$-\frac{\ddot{x}}{x} = \pm q_1^2 \; ; \qquad -\frac{\ddot{y}}{y} = \pm q_2^2 \qquad \text{(i)}$$

and the equations

$$-\frac{\ddot{x}}{x} = \pm\frac{q_1^2}{\phi^4} - \frac{\ddot{\phi}}{\phi} \; ; \qquad -\frac{\ddot{y}}{y} = \pm\frac{q_2^2}{\phi^4} - \frac{\ddot{\phi}}{\phi} \; , \qquad \text{(ii)}$$

where ϕ is an arbitrary function of time. As is well known, equations (i) define Lissajous' curves in the (x, y) plane, and it is seen that equations (ii) define the same Lissajous curves in the (ξ, η) plane where $\xi = x/\phi$ and $\eta = y/\phi$.

We then find that

$$\left.\begin{aligned} \xi_0 &= -\frac{\delta_{10}q_1 - \delta_{20}\beta}{\beta^2 - q_1^2}\sin(q_1 t + \iota_0)\,, \\ \eta_0 &= +\frac{\delta_{20}q_1 - \delta_{10}\beta}{\beta^2 - q_1^2}\cos(q_1 t + \iota_0)\,. \end{aligned}\right\} \tag{1727}$$

Comparing equations (1725) and (1727), we see that *the principal axes of the* (ξ_0, η_0) *ellipse are along the same directions as those of the* δ*-ellipse.*

We may notice the following special cases: (*a*) If $\delta_{10} = \delta_{20}$, then both the δ-locus and the (ξ_0, η_0) locus are circles. (*b*) If

$$\delta_{20}q_1 = \delta_{10}\beta \qquad \text{or} \qquad \delta_{10}q_1 = \delta_{20}\beta\,, \tag{1728}$$

then the (ξ_0, η_0) locus is a straight line along the ξ or the η-axis, respectively.[85]

It is seen that, according to equations (1725) and (1727), the (ξ_0, η_0) locus is uniquely determined by the δ-locus and the value of β.

Case (ii): The δ*-locus an ellipse, and* $|q_1| = |\beta|$.—The analysis of the preceding paragraphs does not apply to this case. When $|q_1| = |\beta|$, equations (1722) take the form

$$\frac{\delta_1}{\phi} = \delta_{10}\sin(\beta t + \iota_1)\,; \qquad \frac{\delta_2}{\phi} = \delta_{20}\sin(\beta t + \iota_2)\,. \tag{1729}$$

Substituting the foregoing expressions for δ_1/ϕ and δ_2/ϕ in equations (1713) and (1714) and carrying out the necessary integrations, we find that

$$\left.\begin{aligned} \xi_0 = \tfrac{1}{2}\delta_{10}&\left[t\sin(\beta t + \iota_1) - \frac{1}{2\beta}\cos(\beta t + \iota_1)\right] \\ &+ \tfrac{1}{2}\delta_{20}\left[-t\cos(\beta t + \iota_2) + \frac{1}{2\beta}\sin(\beta t + \iota_2)\right]\,, \\ \eta_0 = \tfrac{1}{2}\delta_{10}&\left[t\cos(\beta t + \iota_1) - \frac{1}{2\beta}\sin(\beta t + \iota_1)\right] \\ &+ \tfrac{1}{2}\delta_{20}\left[t\sin(\beta t + \iota_2) - \frac{1}{2\beta}\cos(\beta t + \iota_2)\right]\,. \end{aligned}\right\} \tag{1730}$$

[85] Since for a positive sense of the rotation Θ_0, given by equation (1590), β is negative, the relations (1728) imply that in both the cases considered the δ-locus is described in a sense opposite to that of Θ_0.

As in the preceding case, the expressions for ξ_0 and η_0 take simpler forms when the ξ-axis and the η-axis are so chosen that they are in the directions of the principal axes of the δ-ellipse. If this is achieved by an appropriate rotation about the z-axis, equations (1729) will take the forms

$$\frac{\delta_1}{\phi} = \delta_{10} \cos (\beta\iota + \iota_0) \; ; \qquad \frac{\delta_2}{\phi} = \delta_{20} \sin (\beta\iota + \iota_0) \; . \qquad (1731)$$

The corresponding expressions for ξ_0 and η_0 are readily obtained (cf. Eq. [1726]). We find that

$$\left.\begin{aligned} \xi_0 &= \tfrac{1}{2}(\delta_{10} - \delta_{20})\iota \cos (\beta\iota + \iota_0) + \frac{1}{4\beta}(\delta_{10} + \delta_{20}) \sin (\beta\iota + \iota_0) \; , \\ \eta_0 &= -\tfrac{1}{2}(\delta_{10} - \delta_{20})\iota \sin (\beta\iota + \iota_0) - \frac{1}{4\beta}(\delta_{10} + \delta_{20}) \cos (\beta\iota + \iota_0) \; . \end{aligned}\right\} \qquad (1732)$$

Equations (1732) represent a general class of spirals. The simplest example of these spirals arises when

$$\delta_{10} = -\delta_{20} \; . \qquad (1733)$$

According to (1731), the δ-locus is now a circle; and, if β is positive, the δ-circle is described in the negative clockwise direction, i.e., in the same sense as the rotation Θ_0 arising from the term $\boldsymbol{r} \times \boldsymbol{\beta}$ (cf. Eq. [1590]).

From equations (1732) it now follows that

$$\xi_0 = \delta_{10}\iota \cos (\beta\iota + \iota_0) \; ; \qquad \eta_0 = -\delta_{10}\iota \sin (\beta\iota + \iota_0) \; . \quad (1734)$$

Thus, in this case, *the* (ξ_0, η_0) *locus reduces to the spiral of Archimedes*, described in the positive or the negative sense according as β is negative or positive (see Fig. 11).

Again, if $\delta_{10} = \delta_{20}$, the (ξ_0, η_0) locus is a circle described in the positive or the negative sense according as β is positive or negative.

In general, equations (1732) represent loci belonging to the same general class as the *trochoidal curves* associated with the rolling of a straight line on a circle[86] (see Fig. 12).

[86] A useful reference for information concerning these transcendental curves is H. Lamb, *Infinitesimal Calculus*, chaps. ix and x, pp. 284–367, Cambridge, England, 1938.

Case (iii): The δ-locus a straight line ($q_1 = 0$).—If $q_1 = 0$, the solutions for the δ's according to (1719) cease to be valid. Under these circumstances we have (Eqs. [1700], [1701], and [1703])

$$\frac{\ddot{\delta}_1}{\delta_1} = \frac{\ddot{\delta}_2}{\delta_2} = \frac{\ddot{\phi}}{\phi} . \tag{1735}$$

From the foregoing equations we readily conclude that (cf. Eq. [1381])

$$\frac{\delta_1}{\phi} = \delta_{10}(\iota + \iota_1) \; ; \qquad \frac{\delta_2}{\phi} = \delta_{20}(\iota + \iota_2) \, , \tag{1736}$$

where δ_{10}, δ_{20}, ι_1, and ι_2 are constants of integration, and ι has the

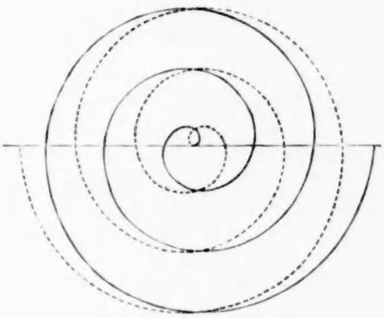

FIG. 11

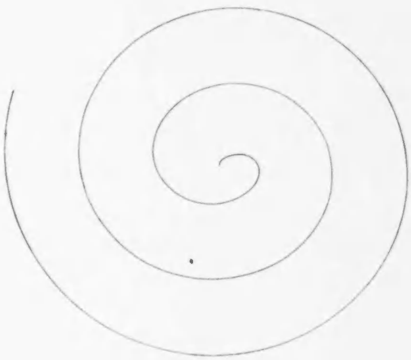

FIG. 12

FIG. 11.—An Archimedean spiral

FIG. 12.—A special case ($\iota_0 = -\pi/2$; $\delta_{10} = 2$; $\delta_{20} = 0$) of the general family of spirals (1732).

same meaning as hitherto. Thus, *when* $q_1 = 0$, *the δ-locus is a straight line.*

Substituting for δ_1/ϕ and δ_2/ϕ according to (1736) in equations (1713) and (1714) and carrying out the necessary integrations, we find

$$\xi_0 = \frac{\delta_{10}}{\beta^2} + \frac{\delta_{20}}{\beta}(\iota + \iota_2) \; ; \qquad \eta_0 = \frac{\delta_{20}}{\beta^2} - \frac{\delta_{10}}{\beta}(\iota + \iota_1) \, . \tag{1737}$$

Hence, the (ξ_0, η_0) locus is a straight line which is at right angles to the δ straight line.

Case (iv): The δ-locus a hyperbola.—If the term q_1^2/ϕ^4 occurs with a negative sign in equation (1718), the expressions for δ_1/ϕ and δ_2/ϕ involve hyperbolic functions; and, according to equation (1719), we can write, quite generally,

$$\left.\begin{aligned} \frac{\delta_1}{\phi} &= \delta_{11} \sinh q_1\iota + \delta_{12} \cosh q_1\iota \, , \\ \frac{\delta_2}{\phi} &= \delta_{21} \sinh q_1\iota + \delta_{22} \cosh q_1\iota \, , \end{aligned}\right\} \tag{1738}$$

where $\delta_{11}, \ldots, \delta_{22}$ are arbitrary constants. By eliminating ι between the foregoing expressions, it is seen that the δ-locus is a hyperbola. Consequently, if by an appropriate rotation about the z-axis we arrange that the ξ- and the η-axes are along the principal axes of the δ-hyperbola, the equations (1738) will take the simpler forms

$$\frac{\delta_1}{\phi} = \delta_{10} \cosh (q_1\iota + \iota_0) \; ; \qquad \frac{\delta_2}{\phi} = \delta_{20} \sinh (q_1\iota + \iota_0) \, . \tag{1739}$$

Using these expressions for δ_1/ϕ and δ_2/ϕ in equations (1713) and (1714), we find that

$$\left.\begin{aligned} \xi_0 &= \frac{\delta_{10} q_1 + \delta_{20}\beta}{\beta^2 + q_1^2} \sinh (q_1\iota + \iota_0) \, , \\ \eta_0 &= \frac{\delta_{20} q_1 - \delta_{10}\beta}{\beta^2 + q_1^2} \cosh (q_1\iota + \iota_0) \, . \end{aligned}\right\} \tag{1740}$$

Thus, the (ξ_0, η_0) locus is a hyperbola the axis of which coincides with the conjugate axis of the δ-hyperbola.

We may notice that we could have derived the foregoing expressions for ξ_0 and η_0 directly from (1727) according to the following scheme of transformations:

$$\left.\begin{aligned} & q_1 \to i q_1 \; ; \qquad \delta_{10} \to \delta_{10} \; ; \quad \delta_{20} \to - i\delta_{20} \, , \\ & \cos (q_1\iota + \iota_0) \to \cosh (q_1\iota + \iota_0) \; ; \\ & \sin (q_1\iota + \iota_0) \to i \sinh (q_1\iota + \iota_0) \, . \end{aligned}\right\} \tag{1741}$$

II. *The solution for* ζ_0.—We have to distinguish between the following three cases:

(i) If q_2^2/ϕ^4 occurs with a positive sign in equation (1718), the solution for δ_3/ϕ involves the circular functions, and, according to (1719), we have

$$\frac{\delta_3}{\phi} = \delta_{30} \sin (q_2 \iota + \iota_3) . \tag{1742}$$

From equation (1715) it now follows that

$$\zeta_0 = -\frac{\delta_{30}}{q_2} \cos (q_2 \iota + \iota_3) . \tag{1743}$$

(ii) If $q_2 = 0$, the solution for δ_3 according to equation (1719) is no longer valid. Instead, we have (cf. Eq. [1736])

$$\frac{\delta_3}{\phi} = \delta_{30}(\iota + \iota_3) . \tag{1744}$$

Accordingly, we now have

$$\zeta_0 = \delta_{30}(\tfrac{1}{2}\iota^2 + \iota_3 \iota) . \tag{1745}$$

(iii) If q_2^2/ϕ^4 occurs with a negative sign in equation (1718), the corresponding solution for δ_3/ϕ involves hyperbolic functions; and, according to (1719), we can write quite generally that

$$\frac{\delta_3}{\phi} = \delta_{31} \cosh q_2 \iota + \delta_{32} \sinh q_2 \iota , \tag{1746}$$

where δ_{31} and δ_{32} are arbitrary constants. The corresponding solution for ζ_0 is readily seen to be

$$\zeta_0 = \frac{1}{q_2} (\delta_{31} \sinh q_2 \iota + \delta_{32} \cosh q_2 \iota) . \tag{1747}$$

III. *The solution for the density function* σ.—According to the general relations derived in § 70, the solution for the density function σ

is directly expressible in terms of the solution of the nonhomogeneous differential equation for $\mathfrak{V}$. Thus (cf. Eq. [1609]),

$$\left.\begin{aligned}\tfrac{1}{2}\sigma = \phi^2\mathfrak{V} + \tfrac{1}{2}\phi\ddot{\phi}r^2 - \frac{1}{2\phi^2}\,|\,\boldsymbol{r}\times\boldsymbol{\beta}+\boldsymbol{\delta}\,|^2 \\ + \phi\frac{d}{dt}\left(\frac{\boldsymbol{\delta}}{\phi}\right)\cdot\boldsymbol{r} + \text{constant}\ ;\end{aligned}\right\}\qquad(1748)$$

or, substituting for $\mathfrak{V}$ our present solution (Eqs. [1711], [1718], and [1721]),

$$\left.\begin{aligned}\tfrac{1}{2}\sigma = \pm\tfrac{1}{2}q_1^2(\xi^2+\eta^2) \pm \tfrac{1}{2}q_2^2\zeta^2 \\ + \mathfrak{V}^*\{(\xi-\xi_0)^2+(\eta-\eta_0)^2\,;\ \vartheta+\beta\iota\,;\ \zeta-\zeta_0\} \\ - \frac{1}{2}\left|\boldsymbol{\rho}\times\boldsymbol{\beta}+\frac{\boldsymbol{\delta}}{\phi}\right|^2 + \frac{d}{d\iota}\left(\frac{\boldsymbol{\delta}}{\phi}\right)\cdot\boldsymbol{\rho} + \text{constant}\ ,\end{aligned}\right\}\qquad(1749)$$

where we have used $\boldsymbol{\rho}$ to denote the vector (ξ, η, ζ). Remembering that, in the co-ordinate system used, $\boldsymbol{\beta} = (0, 0, \beta)$, it is seen that equation (1749) can be expressed as

$$\sigma = 2\mathfrak{V}^* + \sigma_1 + \sigma_2 + \text{constant}\ , \qquad(1750)$$

where

$$\left.\begin{aligned}\tfrac{1}{2}\sigma_1 = \tfrac{1}{2}(\pm q_1^2-\beta^2)(\xi^2+\eta^2) + \beta\left(\xi\frac{\delta_2}{\phi}-\eta\frac{\delta_1}{\phi}\right) \\ + \xi\frac{d}{d\iota}\left(\frac{\delta_1}{\phi}\right) + \eta\frac{d}{d\iota}\left(\frac{\delta_2}{\phi}\right) - \frac{1}{2}\left(\frac{\delta_1}{\phi}\right)^2 - \frac{1}{2}\left(\frac{\delta_2}{\phi}\right)^2\end{aligned}\right\}\qquad(1751)$$

and

$$\tfrac{1}{2}\sigma_2 = \pm\tfrac{1}{2}q_2^2\zeta^2 + \zeta\frac{d}{d\iota}\left(\frac{\delta_3}{\phi}\right) - \frac{1}{2}\left(\frac{\delta_3}{\phi}\right)^2 . \qquad(1752)$$

In considering the explicit expressions for σ_1 and σ_2, we should distinguish between the various cases treated separately in the subsections I and II above.

(A) *Solutions for* σ_1.—*First*, consider the case when the δ-locus is an ellipse and $|q_1| \neq |\beta|$. We then have (cf. Eq. [1719])

$$\frac{\delta_1}{\phi} = \delta_{10}\sin(q_1\iota+\iota_1)\ ; \qquad \frac{\delta_2}{\phi} = \delta_{20}\sin(q_1\iota+\iota_2)\ . \qquad(1753)$$

Substituting the foregoing expressions for δ_1/ϕ and δ_2/ϕ in equation (1751), we find, after some lengthy reductions involving the use of the solutions for ξ_0 and η_0 appropriate to this case (Eq. [1724]), that

$$\left.\begin{aligned}\sigma_1 = (q_1^2 - \beta^2)[(\xi - \xi_0)^2 + (\eta - \eta_0)^2]\\ - \frac{1}{q_1^2 - \beta^2}[q_1^2(\delta_{10}^2 + \delta_{20}^2) + 2\delta_{10}\delta_{20}q_1\beta \sin(\iota_2 - \iota_1)]\,.\end{aligned}\right\} \quad (1754)$$

If the ξ and the η axes are arranged to lie along the principal axes of the δ-ellipse, the expression for σ_1 becomes (see Eq. [1726])

$$\left.\begin{aligned}\sigma_1 = (q_1^2 - \beta^2)[(\xi - \xi_0)^2 + (\eta - \eta_0)^2]\\ - \frac{1}{q_1^2 - \beta^2}[q_1^2(\delta_{10}^2 + \delta_{20}^2) - 2\delta_{10}\delta_{20}q_1\beta]\,.\end{aligned}\right\} \quad (1755)$$

Second, consider the case when the δ-locus is an ellipse and $|q_1| = |\beta|$. Using equations (1731) for the δ's, we find that

$$\left.\begin{aligned}\sigma_1 = -2\beta(\delta_{10} - \delta_{20})[\xi \sin(\beta\iota + \iota_0) + \eta\cos(\beta\iota + \iota_0)]\\ - \delta_{10}^2\cos^2(\beta\iota + \iota_0) - \delta_{20}^2\sin^2(\beta\iota + \iota_0)\,.\end{aligned}\right\} \quad (1756)$$

Using the solutions for ξ_0 and η_0 appropriate to this case (Eq. [1732]), the foregoing expression for σ_1 can be reduced to

$$\left.\begin{aligned}\sigma_1 = -\frac{2\beta(\delta_{10} - \delta_{20})}{\sqrt{(\xi - \xi_0)^2 + (\eta - \eta_0)^2}}\sin(\beta\iota + \vartheta + \iota_0)\\ - \tfrac{1}{2}(\delta_{10}^2 + \delta_{20}^2)\,.\end{aligned}\right\} \quad (1757)$$

Third, consider the case when the δ-locus is a straight line. Using equations (1736) and (1737), we find that

$$\left.\begin{aligned}\sigma_1 = -\beta^2[(\xi - \xi_0)^2 + (\eta - \eta_0)^2] +\\ \frac{1}{\beta^2}[\delta_{10}^2 + \delta_{20}^2 + 2\beta\delta_{10}\delta_{20}(\iota_2 - \iota_1)]\,.\end{aligned}\right\} \quad (1758)$$

Fourth, we have finally to consider the case when the δ-locus is a hyperbola. The appropriate solution for σ_1 can be readily ob-

tained from equation (1755) by transforming it according to the scheme (1741). We find that

$$\left.\begin{aligned}\sigma_1 = &-(q_1^2+\beta^2)[(\xi-\xi_0)^2+(\eta-\eta_0)^2]\\ &+\frac{1}{q_1^2+\beta^2}[q_1^2(\delta_{20}^2-\delta_{10}^2)-2\delta_{10}\delta_{20}q_1\beta]\,.\end{aligned}\right\} \qquad (1759)$$

(B) *Solutions for σ_2.*—We have to distinguish now between the three cases considered in section II above. For these three cases we respectively find that

$$\sigma_2 = \left\{\begin{aligned} &q_2^2(\zeta-\zeta_0)^2-\delta_{30}^2\,,\\ &2\delta_{30}(\zeta-\zeta_0)-\delta_{30}^2\iota_3^2\,,\\ &-q_2^2(\zeta-\zeta_0)^2-\delta_{31}^2+\delta_{32}^2\,.\end{aligned}\right\} \qquad (1760)$$

We notice at once a remarkable fact about our solutions for σ_1 and σ_2, namely, that *they are expressible as functions of the same first-integrals of which $\mathfrak{B}^*$ is a function.* Consequently, in equation (1750) we can absorb σ_1 and σ_2 in $\mathfrak{B}^*$ and write quite generally that

$$\sigma \equiv \sigma\{(\xi-\xi_0)^2+(\eta-\eta_0)^2\,;\ \vartheta+\beta\iota\,;\ \zeta-\zeta_0\}\,. \qquad (1761)$$

Thus, σ is a function only of the arguments specified.

Finally, if we specialize our general spherical distribution of residual velocities to a Maxwellian distribution, then the number of stars per unit volume, $\mathfrak{N}$, is related to σ by (Eq. [1644])

$$\mathfrak{N} = \frac{\pi^{3/2}}{\phi^3}e^{-\sigma}\,. \qquad (1762)$$

Hence, according to (1761), we can write

$$\mathfrak{N} = \frac{1}{\phi^3}\mathfrak{N}_1\{(\xi-\xi_0)^2+(\eta-\eta_0)^2\,;\ \vartheta+\beta\iota\,;\ \zeta-\zeta_0\}\,, \qquad (1763)$$

where $\mathfrak{N}_1$ is a function only of the arguments specified.

75. *An interpretation of the spiral structure in nebulae.*—Before we proceed to consider applications of the solutions obtained in §§ 73

and 74 to an interpretation of the spiral structure in nebulae, we shall briefly restate the assumptions that have been made.

Our fundamental assumption has, of course, been the restriction to a spherical distribution of the residual velocities:

$$\Psi \equiv \Psi[a\{(U - U_0)^2 + (V - V_0)^2 + (W - W_0)^2\} + \sigma] , \quad (1764)$$

where a, U_0, V_0, W_0, and σ are all functions of x, y, z, and t, arbitrary in the first instance. Further, Ψ is itself an arbitrary function of the argument specified. This assumption concerning Ψ has been made primarily with a view to discovering the most general types of potential and density distributions that are consistent with our fundamental kinematical postulates. But this restriction to a spherical distribution of residual velocities may appear serious, as it ignores the phenomenon of star-streaming. However, according to the ***principle of superposition of stellar systems*** considered in Part XIV, it is formally a simple matter to include star-streaming in the sense of Kapteyn.[87]

The assumption of a spherical distribution of velocities (1764), together with the equation of continuity, allows us to conclude that (i)

$$a = \phi^2 , \quad (1765)$$

where ϕ is an arbitrary function of time, (ii)

$$(U_0, V_0, W_0) = \frac{\dot{\phi}}{\phi}\boldsymbol{r} + \frac{1}{\phi^2}(\boldsymbol{r} \times \boldsymbol{\beta} + \boldsymbol{\delta}) , \quad (1766)$$

where $\boldsymbol{\beta}$ is a constant vector and $\boldsymbol{\delta}$ a function of time, and (iii) $\mathfrak{V}$ satisfies a single nonhomogeneous linear partial differential equation (Eq. [1597]). The homogeneous equation associated with (1597) has been solved for its most general solution and a particular integral has been found which has the same form as that which is known to give the most general particular integral for the case $\boldsymbol{\delta} \equiv 0$. The restriction to this special form of the particular integral has one advantage, namely, that $\boldsymbol{\delta}$ is now no longer an arbitrary function but

[87] For further details on how this can be done see § 81.

is related to ϕ according to certain transcendental relations. We shall see later (§ 76) that, if $\boldsymbol{\delta}$ remains arbitrary, then none of the features to be discussed are lost—indeed, they are further generalized.

Thus, the characteristics of stellar systems we are now going to discuss arise from only *two* assumptions: the physical assumption of a spherical distribution of the residual velocities and the mathematical restriction to a particular integral of a nonhomogeneous equation of a certain form.[88]

Under the circumstances described, the potential $\mathfrak{B}$ is given by[89]

$$\left.\begin{aligned}\mathfrak{B} = \frac{1}{2}\left(\pm\frac{q_1^2}{\phi^2} - \phi\ddot{\phi}\right)(\xi^2 + \eta^2) + \frac{1}{2}\left(\pm\frac{q_2^2}{\phi^2} - \phi\ddot{\phi}\right)\zeta^2 \\ + \frac{1}{\phi^2}\mathfrak{B}^*\{(\xi - \xi_0)^2 + (\eta - \eta_0)^2 ; \vartheta + \beta\iota ; \zeta - \zeta_0\} ,\end{aligned}\right\} \quad (1767)$$

where

$$x = \xi\phi ; \qquad y = \eta\phi ; \qquad z = \zeta\phi ; \qquad \iota = \int_{t_0}^{t}\frac{dt}{\phi^2} . \qquad (1768)$$

Further, in equation (1767) q_1 and q_2 are two arbitrary real numbers; ξ_0, η_0, and ζ_0 are certain functions of ι, $\vartheta = \tan^{-1}(\eta - \eta_0)/(\xi - \xi_0)$, and $\mathfrak{B}^*$ is an arbitrary function of the arguments specified. The explicit expressions for ξ_0, η_0, and ζ_0 are given in § 74. As we have shown in § 74, ξ_0 and η_0 define a locus in the (ξ, η) plane which can be an ellipse, an Archimedean spiral (or a generalization of it according to Eq. [1732]), a straight line, or a hyperbola. Finally, for the density function σ we have (cf. Eqs. [1750]–[1752], [1755], and [1757]–[1760])

$$\sigma = 2\mathfrak{B}^* + \sigma_1 + \sigma_2 , \qquad (1769)$$

[88] It should perhaps be emphasized that the physical assumption can be generalized to include star-streaming in the sense of Kapteyn. Also, the restriction to a certain special form of the particular integral has been made primarily with a view to making the discussion more concrete and specific. As we shall see in § 76, the consideration of the most general particular integral of the equation (1597) serves only to generalize still further the scheme developed in this section.

[89] From now on it is assumed that the co-ordinate system is so chosen that the z-axis is along the vector $\boldsymbol{\beta}$.

where σ_1 and σ_2 are certain expressions which are functions of the same first-integrals which $\mathfrak{B}^*$ involves. Consequently, we can write

$$\sigma \equiv \sigma\{(\xi - \xi_0)^2 + (\eta - \eta_0)^2 \,;\, \vartheta + \beta\iota \,;\, \zeta - \zeta_0\} \,. \tag{1770}$$

(A) *The simple theory.*—We shall begin our discussion of the density function σ by considering the simplest case, namely, when the system is characterized by no translational motions (i.e., $\boldsymbol{\delta} \equiv 0$). Then, according to equations (1713) and (1714),

$$\xi_0 = \eta_0 = \zeta_0 \equiv 0 \tag{1771}$$

and

$$\sigma \equiv \sigma\left(\frac{\bar{\omega}}{\phi},\, \theta + \beta\int^t \frac{dt}{\phi^2},\, \frac{z}{\phi}\right). \tag{1772}$$

Equation (1770) enables us to describe the trajectories of points of constant σ. These trajectories are, in fact, defined parametrically by the relations

$$\bar{\omega} = c_1\phi \,; \qquad \theta = c_2 - \beta\iota \,; \qquad z = c_3\phi \,, \tag{1773}$$

where c_1, c_2, and c_3 are constants. As we have already seen in § 72, the trajectories (1773) are spirals described outward in the direction of rotation if $\dot{\phi}$ is positive and described inward (also in the direction of rotation) if $\dot{\phi}$ is negative (see Figs. 9*a* and 9*b*). Further, the motions of density points along these trajectories are governed directly by the motions of the local centroids; i.e., if $d\bar{\omega}$, $d\theta$, and dz are the increments in $\bar{\omega}$, θ, and z due to an increment dt in t, then (cf. Eqs. [1653] and [1662])

$$d\bar{\omega} = \Pi_0 dt \,; \qquad \bar{\omega}d\theta = \Theta_0 dt \,; \qquad dz = Z_0 dt \,. \tag{1774}$$

Accordingly, the angle α which a spiral orbit makes with the transverse direction is given by

$$\alpha = \tan^{-1}\frac{\Pi_0}{\Theta_0} \,; \tag{1775}$$

if this angle is a constant, the orbit reduces to a logarithmic spiral. Also, for the case under consideration (cf. Eq. [1637])

$$\Pi_0 = \frac{\dot{\phi}}{\phi}\,\bar{\omega}\,; \qquad \Theta_0 = -\frac{\beta}{\phi^2}\,\bar{\omega}\,; \qquad Z_0 = \frac{\dot{\phi}}{\phi}\,z\,. \tag{1776}$$

Hence, the rotational velocity increases linearly with the distance from the axis of rotation. Further,

$$\frac{Z_0}{\Pi_0} = \frac{z}{\bar{\omega}}\,. \tag{1777}$$

Consequently, if the stellar system is a very flat one, the motion in the z-direction is only a small fraction of the motion in the radial direction except in the neighborhood of the center. Since the radial motions are themselves only small fractions of the rotational velocities, it is clear that the motions perpendicular to the galactic plane are quite negligible. Finally, we may notice that if we have an initial distribution of matter along a spiral orbit, then it remains along this orbit. Thus, the spiral structure has a certain degree of *permanence.*

The foregoing description of the essential features of the stellar system we are considering in its simplest form (namely, when $\boldsymbol{\delta} \equiv 0$) strongly suggests the possibility of using this theory as a basis for interpreting the spiral structure in nebulae. In suggesting this, we are guided by the following circumstances:

(i) The occurrence of spiral orbits for points of constant relative density;[90] (ii) the degree of permanence which is attributed to the spiral structure; and (iii) the linearity of Θ_0 with distance which is in approximate agreement with the general "velocity-curve" derived by Babcock for the Andromeda nebula. Apart from these considerations, the strongest support for the suggestion comes, however, from the close similarity between the characteristics we have derived on the basis of our theory and the general features that are present in Lindblad's theory of spiral structure. In Lindblad's the-

[90] For the case $\Psi(Q+\sigma) = e^{-Q-\sigma}$ the density of stars $\mathfrak{N}$ is given by $\pi^{3/2}e^{-\sigma}/\phi^3(t)$. Consequently, for this important case, the dependence of $\mathfrak{N}$ on the spatial co-ordinates is governed by σ. We shall therefore refer to $e^{-\sigma}$ as the *relative density.*

ory the emphasis is on the orbits described by individual stars; instead, we consider the orbits of constant relative density; in both cases spiral orbits are derived. Again, there is agreement between Lindblad's theory and the present one in that in both the linearity of the rotational velocity with distance is found. Finally, if $\dot{\phi}$ is positive, there is also agreement between the two theories in the prediction concerning the direction of rotation relative to the winding of the spiral: the motion is outward along the spiral in the direction of rotation. Formally, in our theory we can get the contrary sense between the direction of rotation and the winding of the spiral if $\dot{\phi}$ is negative; but the assumption $\dot{\phi} < 0$ would imply (i) a contraction and (ii) an increase in the dispersion of the residual velocities with time and both these consequences of $\dot{\phi} < 0$ appear to be unlikely.

It is now seen that the whole group of problems connected with the tilt of the Andromeda nebula relative to the line of sight is again brought to the center of attention. The final settlement of this question is clearly a matter of the utmost importance. While there is considerable divergence of views on this matter, we shall accept Lindblad's verdict that the tilt is such as to make the prediction based on his theory to be in agreement with the measured radial velocities. In our case this assumption would imply that $\dot{\phi}$ is positive.

In spite of this general agreement between Lindblad's theory and our own, the fundamental differences in the two approaches should not be overlooked. In Lindblad's work the attention is always focused on the instability of the circular orbits which arises whenever there is a sufficiently steep density gradient.[91] The consequences of this instability are then further discussed, but largely in a qualitative manner. While recognizing the importance of such considerations, the vagueness and the lack of clarity which results are, in the writer's view, due essentially to the circumstances of having to describe nonsteady-state phenomena without having an adequate theory in terms of which to interpret such phenomena. It is therefore satisfactory that in our theory the center of attention is brought precisely to this intimate connection between the spiral phenomenon,

[91] Lindblad's criterion for the instability of the circular orbits is $\bar{\rho} - \rho_0 > \omega_c^2/2\pi G$ (see *Stockholms observatoriums annaler*, **12**, No. 4, p. 23).

on the one hand, and the fundamentally nonsteady character of the systems involved, on the other. Further, in contrast to Lindblad's theory the assumptions that have been made are few and well defined. But it is perhaps a disadvantage of the present treatment of spiral phenomenon that a clear separation had to be made between the methods for dealing with ellipsoidal systems and those for dealing with spiral systems. Indeed, at the present stage of development of the theory the two methods appear to be mutually exclusive. But we may hope that this gap will eventually be bridged.

There is one further point in the simple theory we have described which requires some further consideration and that is the degree of openness of the spiral predicted. According to equation (1775), the angle which the spiral orbit makes with the transverse direction is given by $\tan^{-1} \Pi_0/\Theta_0$. This angle can be estimated as follows: The existence of a Π_0 of the form (1776) implies a K-term proportional to distance. The probable value of such a K-term is not likely to exceed 7.5 km/sec/1000 parsecs. At a distance of 8000 parsecs from the center this requires a value of $\Pi_0 = 60$ km/sec. Also, at a distance of 8000 parsecs from the center we may expect a rotational velocity of about 250 km/sec. Hence,

$$\alpha \simeq \tan^{-1} 0.24 \simeq 14^\circ . \tag{1778}$$

However, for most well-resolved open spirals the value of α appears to be considerably in excess of the amount (1778), being more nearly 45°. On the other hand, for "compact" spirals such as NGC 7217, the angle α is seen to be about 10°. Thus, it appears that the special case of the general theory we have outlined above is probably applicable only to compact spirals like NGC 7217 and that well-resolved open spirals like M 101 are beyond the range of this simple theory.[92] Consequently, an interpretation of these objects, as well as of the barred and other spirals, is to be sought in the characteristics of the general solution (1770) for the density function σ.

[92] It would appear that an interpretation of really open spirals like M 101 or NGC 6946 is also beyond the range of Lindblad's theory, for these objects are so far removed from elliptical systems that considerations essentially connected with the instability of circular orbits in flattened spheroidal systems are no longer of any particular guidance. But, as we shall see presently (see par. [C] in this section), an interpretation of open spirals appears to be entirely within the scope of our theory.

(B) *The general characteristics of the density function, σ.*—According to the general solution (1770), the trajectory of a point of constant σ in the (ξ, η, ζ) space is described parametrically by the relations

$$\left.\begin{aligned}(\xi - \xi_0)^2 + (\eta - \eta_0)^2 &= c_1^2 ,\\ \tan^{-1} \frac{\eta - \eta_0}{\xi - \xi_0} &= c_2 - \beta\iota ,\end{aligned}\right\} \tag{1779}$$

and

$$\zeta - \zeta_0 = c_3 , \tag{1780}$$

where c_1, c_2, and c_3 are constants. In the foregoing equations, ξ_0, η_0, and ζ_0 are themselves functions of ι. From equations (1779) and (1780) it appears that the motions perpendicular to, and respectively parallel to, the ζ direction can be considered separately.

Now, in the (ξ, η) plane, ξ_0 and η_0 will define a locus which, depending upon circumstances, will be either an ellipse, an Archimedean spiral (or a generalization of it, according to Eq. [1732]), a straight line, or a hyperbola. From the (ξ_0, η_0) locus the trajectory (1779) can be readily derived according to the method outlined in § 73 (pp. 595–596). To be more specific, consider the case when the (ξ_0, η_0) locus is an ellipse and $|q_1| \neq |\beta|$. If the ξ and the η axes are so chosen that they are along the principal axes of the (ξ_0, η_0) ellipse, then (cf. Eq. [1727])

$$\left.\begin{aligned}\xi_0 &= -\frac{\delta_{10}q_1 - \delta_{20}\beta}{\beta^2 - q_1^2} \sin (q_1\iota + \iota_0) ,\\ \eta_0 &= +\frac{\delta_{20}q_1 - \delta_{10}\beta}{\beta^2 - q_1^2} \cos (q_1\iota + \iota_0) .\end{aligned}\right\} \tag{1781}$$

We can regard the ellipse (1781) as being described by a moving representative point P_0 such that equal *eccentric angles* are described in equal intervals of ι, the period of one complete revolution being $2\pi/q_1$.[93] If we now attach to P_0 an "arm" P_0P of length c_1 and allow it to rotate about P_0 at a constant rate $-\beta$, then the locus described by P_0 is the required trajectory. If $\delta_{10} = \delta_{20}$, the (ξ_0, η_0) locus becomes a circle, and the trajectory (1779) reduces to an *epicyclic*,

[93] This, in a measure of "time" according to ι.

since it is now derived as the result of the superposition of two circular motions. Hence, the trajectories described according to equations (1779) and (1781) are generalizations of the epicyclics of ancient Ptolemaic astronomy.

When the basal curve is not an ellipse but one of the other possible curves, the trajectory (1779) can be derived by methods similar to the one described above. Examples of loci described by such methods are illustrated in Figures 13–17. Once the trajectory in the (ξ, η) plane has been obtained, that in the (x, y) plane is readily derived according to the transformation formulae (1768).

The characterization of the motion in the ζ direction is simple. In the (ξ, η, ζ) space the point always remains at a constant distance from ζ_0, where ζ_0 depends on ι according to the relations derived in § 74.

Let us now consider in somewhat greater detail an important feature of the orbits described by points of constant σ in the variables ξ, η, ζ, and ι. In terms of these variables the orbits can be uniquely specified according to equations (1779) and (1780). As we have seen in § 73, the motion in the (ξ, η) plane involves two periods: the period $2\pi/\beta$, with which the orbit is described relative to its varying instantaneous center, and the period associated with the description of the δ-locus (see pp. 601–602). This later period can be real or imaginary according as the δ-locus is an ellipse or a hyperbola. It now appears that the case when the δ-locus is an ellipse is probably more significant physically than when it is a hyperbola. This can be seen as follows:

It will be recalled that the δ-locus arises from the elimination of ϕ between the relations (cf. Eqs. [1700] and [1701])

$$-\frac{\ddot{\delta}_1}{\delta_1} = -\frac{\ddot{\delta}_2}{\delta_2} = \pm\frac{q_1^2}{\phi^4} - \frac{\ddot{\phi}}{\phi}. \tag{1782}$$

From the foregoing equations it follows that (Eqs. [1707] and [1708])

$$\frac{\delta_1}{\phi} = \delta_{10}\left\{\begin{matrix}\sin\\ \sinh\\ \cosh\end{matrix}\right\}(q_1\iota + \iota_1) \tag{1783}$$

and

$$\frac{\delta_2}{\phi} = \delta_{20} \begin{Bmatrix} \sin \\ \sinh \\ \cosh \end{Bmatrix} (q_1\iota + \iota_2) , \tag{1784}$$

where the circular or the hyperbolic functions should be used according as the term q_1^2/ϕ^4 occurs with the positive or the negative sign in (1782). Now, if ϕ were a constant, equations (1782) are precisely the equations for elliptic (or hyperbolic) motion, and in this case physical considerations would allow us to reject the solutions involving the hyperbolic functions. However, we cannot ignore the solutions involving the hyperbolic functions quite as simply when ϕ is not a constant. For, it is very well possible that

$$\underset{t\to\infty}{\text{limit}}\ \iota = \underset{t\to\infty}{\text{limit}} \int_{t_0}^{t} \frac{dt}{\phi^2} \tag{1785}$$

exists. Indeed, physical considerations would rather suggest the convergence of the integral on the right-hand side of equation (1785); for, if the system is one in expansion, ϕ increases with time, and for the convergence of the integral we need only require that ϕ increase more rapidly than $\sqrt{t}$ as $t \to \infty$. On the other hand, since the hyperbolic functions increase very rapidly with increasing values of the argument, it is clear that, when the δ-locus is a hyperbola, the conditions are not favorable for a stable stellar system. Thus, the case when the δ-locus is an ellipse is, from considerations of stability, more important than when the δ-locus is a hyperbola.

It is clear that remarks similar to the foregoing apply equally to the choice of the solution for δ_3 involving the circular function in preference to the solutions involving the hyperbolic functions.

Having thus shown that the physically more significant solutions for the δ's are those which involve the circular functions, we see that stellar systems of the kind we are considering are characterized by three real periods: the periods $2\pi/\beta$, $2\pi/q_1$, and $2\pi/q_2$, associated, respectively, with the rotation $-\beta\bar{\omega}/\phi^2$, the description of the δ-ellipse, and the motions in the ζ-direction. Here, again, the case when the two periods $2\pi/\beta$ and $2\pi/q_1$ are equal or nearly equal is probably more important than the other cases. As we shall see

presently, the case when these two periods are equal leads to a fairly straightforward interpretation of the well-resolved open spirals, while the case when the two periods are nearly equal allows the possibility of interpreting barred spirals. Finally, the more peculiar spirals appear to be explicable on the basis of the more general cases when the two periods $2\pi/\beta$ and $2\pi/q_1$ are widely different.

(C) *An interpretation of open well-resolved spirals.*—As we have already indicated, the case of greatest interest arises when the two periods associated with the motions in ξ and η are equal, i.e., when $|q_1| = |\beta|$. The (ξ_0, η_0) locus appropriate for this case has been obtained in § 74 and belongs to a general family of spirals. This locus defined by equations (1732) reduces to an Archimedean spiral when the δ-locus is a circle and is further described in the same sense as the rotation $-\beta\bar{\omega}/\phi^2$. We shall consider this case first.

According to equations (1734), we have

$$\xi_0 = \delta_{10}\iota \cos (\beta\iota + \iota_0) \; ; \qquad \eta_0 = -\delta_{10}\iota \sin (\beta\iota + \iota_0) \, . \quad (1786)$$

From the foregoing equations we readily obtain

$$\sqrt{\xi_0^2 + \eta_0^2} = \delta_{10}\iota \; ; \qquad \tan^{-1} \frac{\eta_0}{\xi_0} = -(\beta\iota + \iota_0) \, . \qquad (1787)$$

On the other hand, if θ denotes the azimuthal angle in the (ξ, η) plane,

$$\xi_0 = \sqrt{\xi_0^2 + \eta_0^2} \cos \theta \; ; \qquad \eta_0 = \sqrt{\xi_0^2 + \eta_0^2} \sin \theta \, . \qquad (1788)$$

Combining equations (1787) and (1788), we find

$$\left.\begin{aligned} \beta\iota &= -(\theta + \iota_0) \, , \\ \sqrt{\xi_0^2 + \eta_0^2} &= -\frac{\delta_{10}}{\beta} (\theta + \iota_0) \, . \end{aligned}\right\} \qquad (1789)$$

Remembering that β is negative for a positive sense of rotation, we see that equations (1789) define an Archimedean spiral described in the same sense as the direction of Θ_0.

Considering next the trajectory described by a point of constant

relative density σ, it is seen that the angle PP_0Q which the radius vector (ξ, η) makes with the vector (ξ_0, η_0) is given by (see Fig. 13)

$$\widehat{PP_0Q} = \tan^{-1}\frac{\eta - \eta_0}{\xi - \xi_0} - \theta \,, \tag{1790}$$

or, according to equations (1779) and (1789),

$$\widehat{PP_0Q} = c_2 + \iota_0 = \text{constant} \,; \tag{1791}$$

in other words, the "arm" P_0P describing the trajectory is inclined at a constant angle with the radius vector (ξ_0, η_0) describing the

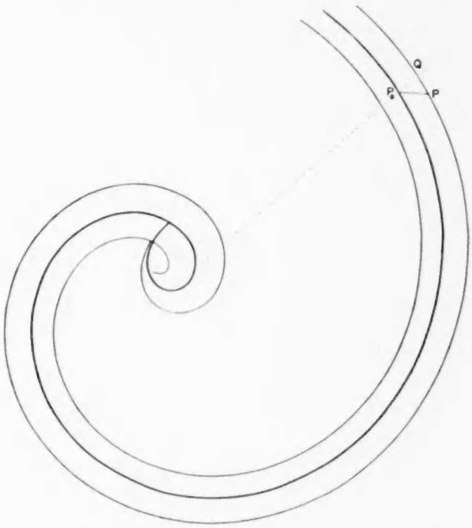

FIG. 13.—Trajectories of points of constant σ in the (ξ, η) plane derived from an Archimedean spiral.

basal Archimedean spiral. Consequently, the equation of the trajectory can be written explicitly as

$$\xi^2 + \eta^2 = \xi_0^2 + \eta_0^2 + c_1^2 + 2c_1\sqrt{\xi_0^2 + \eta_0^2}\cos(c_2 + \iota_0) \,, \tag{1792}$$

or, using equations (1789),

$$\xi^2 + \eta^2 = \frac{\delta_{10}^2}{\beta^2}(\theta + \iota_0)^2 - 2c_1\frac{\delta_{10}}{\beta}(\theta + \iota_0)\cos(c_2 + \iota_0) + c_1^2 \,. \tag{1793}$$

The foregoing equation, together with the relation

$$\theta + \iota_0 = -\beta\iota\,, \tag{1794}$$

defines the trajectory of a point of constant σ in the (ξ, η) plane. Examples of such trajectories are shown in Figure 13.

To obtain the trajectory in the variables x, y, and t corresponding to that defined by the equations (1793) and (1794), we should transform according to the formulae

$$x = \xi\phi\,; \qquad y = \eta\phi\,; \qquad \iota = \int_{t_0}^{t} \frac{dt}{\phi^2}\,. \tag{1795}$$

If ϕ increases with time, the effect of the foregoing transformation will be to "extend" the spiral trajectory in the variables ξ and η into a more open spiral orbit in the variables x and y. Remembering that (1795) transforms the circle, $\xi^2 + \eta^2 = c_1^2$, which occurs in the simple theory (see [A] above) into something like a logarithmic spiral, it is clear that the effect of (1795) on a curve which is already a spiral will be to transform it into a very much more open spiral in the variables x and y. Thus, consider the effect of the transformation (1795) on the basal Archimedean spiral (1789). If, as an illustrative example, we consider the case when ϕ^2 increases linearly with the time (cf. Eq. [1656]), we readily find that the Archimedean spiral (1789) in the variables ξ and η is transformed into the spiral (cf. Eq. [1658])

$$\bar{\omega} = \bar{\omega}_0(\theta + \iota_0)e^{-(A/2\beta)\theta}\,. \tag{1796}$$

Consequently, we can interpret, on our present basis, well-resolved open spirals like M 101.

The motions perpendicular to the galactic plane do not require any special comments except to note that the trajectories of points of constant σ in the z-direction are given by relations of the form

$$z = \phi\left\{c_3 + \frac{\delta_{30}}{q_2}\cos\left(q_2\iota + \iota_3\right)\right\}, \tag{1797}$$

where, according to our remarks in (B), we have chosen the solution for ζ_0 which involves the circular functions. For flat systems the

magnitudes of the motions perpendicular to the galactic plane are quite negligible; but it is of some interest to note that the motions have an oscillatory character.

There is one further point to which we shall draw attention here. In our present interpretation there is no particular reason why we should expect just two spiral arms. Indeed, on the contrary, we should expect several spiral orbits described independently by different groups of stars. An examination of the photographs of the nebulae M 101, NGC 6946 and 2853, and IC 342 shows definite evidence for four or more spiral arms and even indicates the presence of several remote independent spiral orbits.

Finally, when we consider the general (ξ_0, η_0) locus as given by equations (1732), the main features are found to be essentially the same as when the (ξ_0, η_0) locus is an Archimedean spiral. However, the more general spirals (1732) provide just the necessary freedom to account for the divergence (within limits) of forms one encounters among the spiral nebulae.

(D) *An interpretation of barred and other peculiar spirals.*—We have seen how the case $|q_1| = |\beta|$ allows an interpretation of well-resolved open spirals. We shall now consider the general case when the two periods associated with the motions in the (ξ, η) plane are not equal. The (ξ_0, η_0) locus is then an ellipse; and in considering this ellipse according to equation (1727), we can, without loss of generality, choose the arbitrary phase ι_0 appropriately. Let us suppose that $\iota_0 = -\pi/2$. The parametric representation of the (ξ_0, η_0) ellipse becomes

$$\left.\begin{aligned} \xi_0 &= \frac{\delta_{10}q_1 - \delta_{20}\beta}{\beta^2 - q_1^2}\cos q_1\iota\ , \\ \eta_0 &= \frac{\delta_{20}q_1 - \delta_{10}\beta}{\beta^2 - q_1^2}\sin q_1\iota\ . \end{aligned}\right\} \qquad (1798)$$

Examples of trajectories of points of constant σ in the (ξ, η) plane based on the ellipse (1798) are illustrated in Figure 14. When $\delta_{10} = \delta_{20}$ the (ξ_0, η_0) locus is a circle, and the trajectories of points of constant σ in the (ξ, η) plane reduce to the classical epicyclics (Fig. 15). A characteristic of all these trajectories is that they tend to become

circles when the distances from the center are large, compared with the linear dimensions of the basal ellipse. However, in the neighborhood of the ellipse they have a variety of different forms and, depending on circumstances, intersect and interpenetrate them-

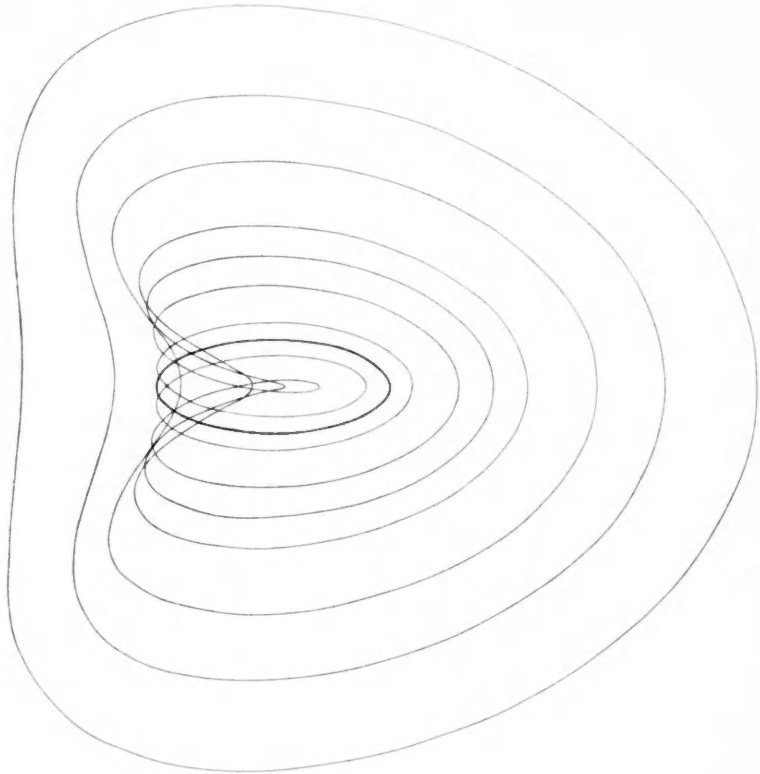

FIG. 14.—Examples of trajectories in the (ξ, η) plane described by points of constant σ derived from an ellipse.

selves. In the (x, y) plane the trajectories therefore tend to become simple spirals of the "elementary" theory (Eqs. [1773]) at large distances and have a wide variety of forms at smaller distances. It thus appears that we may use these different possible forms of the trajectories as a basis for interpreting peculiar nebulae. For example, the trajectories derived from the loci shown in Figure 14 are seen to have a certain resemblance to the shapes of the pecul-

iar nebulae NGC 1559 and 4038 and 4039.[94] Similarly, the trajectories derived from the loci shown in Figures 15 and 16 may be used to interpret other types of peculiar nebulae.

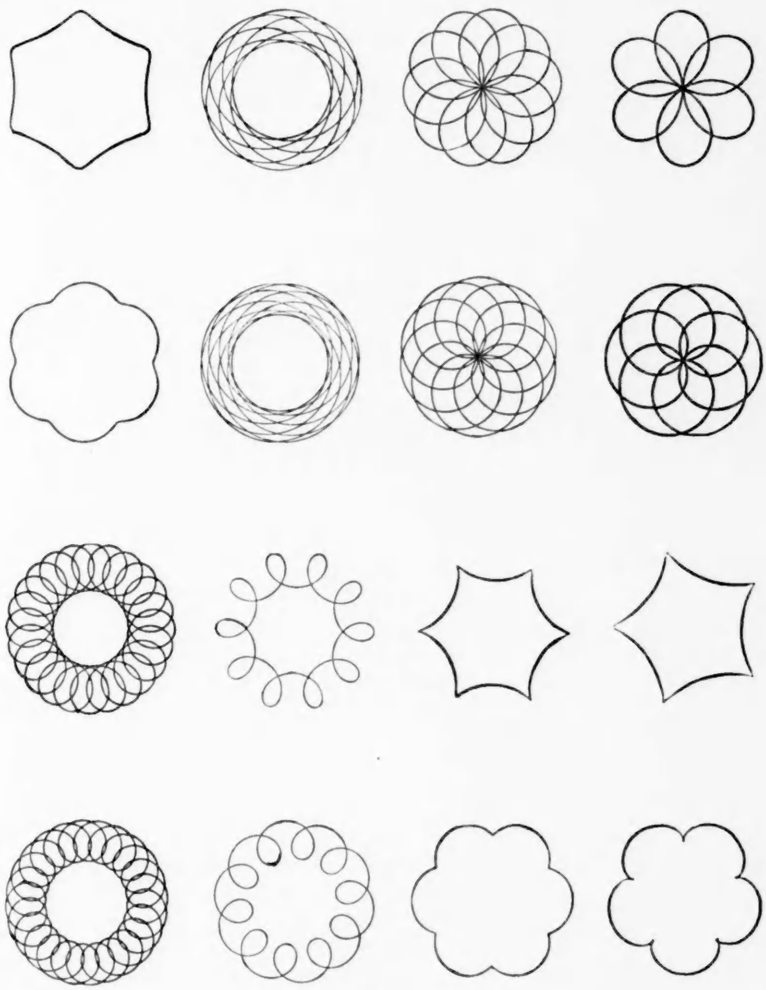

FIG. 15.—Epicyclical orbits

Special interest seems to be attached to cases when the two periods associated with the motions in the (ξ, η) plane are nearly

[94] Photographs of these nebulae have been published by H. Shapley and J. S. Paraskevopoulos, *Proc. Nat. Acad. Sci.*, **26**, 31, 1940 (see esp. Figs. 12 and 20).

equal. Under these circumstances the linear dimensions of the basal ellipse (1798) become very large. Further, the trajectories described with a small "arm" c_1 (cf. Eq. [1779]) will all be near enough to the basal ellipse to give the impression of elliptical orbits. The actual orbits in the (x, y) plane will also be approximately elliptical, particularly if the expansion associated with $\dot{\phi}/\phi$ is very small, compared with the rotational velocities. It is possible that it is these elliptical orbits which give the appearance of a "lens" in the

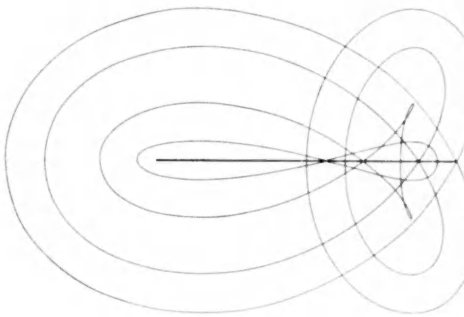

FIG. 16.—Examples of trajectories in the (ξ, η) plane described by points of constant σ derived from a straight line.

early stages of a barred spiral. It is further possible that the "bar" itself may correspond to regions of relatively high density of trajectories inside the basal ellipse. The forms of the nebulae NGC 1300, 1530, 3351, and 5921 are suggestive in this connection.[95]

So far, we have restricted ourselves to only those cases where the solutions for ξ_0, η_0, and ζ_0 involve the circular functions. In (B) we indicated some general reasons why these cases are likely to be more important than the others. However, since the integral defining ι can (and probably does) converge as t tends to infinity, we can still expect rare examples of stellar systems which correspond, for instance, to solutions for ξ_0, η_0, and ζ_0 involving the hyperbolic functions. Thus, the peculiar **V**-shaped nebula NGC 4781[96] suggests by its appearance that we probably have here an example of a system

[95] For photographs of these nebulae see *Handb. d. Ap.*, **5**, No. 1, p. 843, 1933.

[96] For a photograph of this nebula see *Helwan Observatory Bull. No. 22*, Pl. V.

derived on the basis of a (ξ_0, η_0) locus which is a hyperbola (see Fig. 17). Similarly, it is possible that NGC 4656–4657[97] is derived from a basal curve which is a straight line described uniformly with respect to ι (cf. Eqs. [1737] and [1745]); in this case the trajectories are cycloids and the corresponding trochoidal curves.

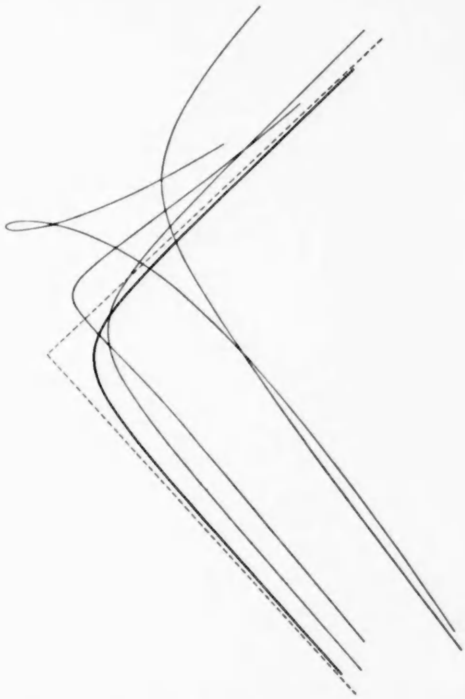

FIG. 17.—Examples of trajectories in the (ξ, η) plane described by points of constant σ derived from a hyperbola.

While the foregoing remarks concerning the peculiar and the barred spirals are to be regarded as largely tentative, they serve to show the wide variety of forms which is possible on the basis of the present theory. It has been generally thought that the widely divergent forms which the spiral nebulae show indicates that no simple theory can account for all of them. But it now appears that

[97] For a photograph of this nebula see *Ap. J.*, **51**, 276, 1920 (Pl. XVI).

this very divergence of forms receives an adequate and straightforward explanation on the basis of our theory, and this is perhaps its most satisfactory feature.

Finally, it may be remarked that the forms of the trajectories of points of constant σ will be further generalized if the basic δ-loci are more general curves than ellipses, straight lines, or hyperbolae; it will be recalled that we were led to these special curves for the δ-loci as the result of our restriction to a special form for the particular integral of the nonhomogeneous partial differential equation for $\mathfrak{V}$. A more general particular integral than the one we have chosen will serve only to generalize still further our present theory (see § 76).

76. *The most general solution of equation (1665) and the general theory of spiral structure.*—So far we have restricted ourselves to the consideration of a certain particular integral of the fundamental differential equation (1665). We shall now obtain the most general solution of this equation and discuss some of its consequences.

Equation (1665) can be re-written as

$$\left.\begin{aligned}\left(\frac{1}{2}\frac{d\kappa}{dt}x+\beta y+\delta_1\right)\frac{\partial F}{\partial x}+\left(\frac{1}{2}\frac{d\kappa}{dt}y-\beta x+\delta_2\right)\frac{\partial F}{\partial y}\qquad\qquad\\+\left(\frac{1}{2}\frac{d\kappa}{dt}z+\delta_3\right)\frac{\partial F}{\partial z}+\kappa\frac{\partial F}{\partial t}+\tfrac{1}{4}(x^2+y^2+z^2)\kappa\frac{d^3\kappa}{dt^3}\\+(x\ddot{\delta}_1+y\ddot{\delta}_2+z\ddot{\delta}_3)\kappa=0\,,\end{aligned}\right\}\quad(1799)$$

where we have used F to denote $\kappa\mathfrak{V}$. Instead of seeking a solution of the foregoing equation directly, we shall try to define it by means of an equation not solved for F:

$$\Omega(F,x,y,z,t)=0\,,\qquad(1800)$$

where the function Ω of the five variables F, x, y, z, and t is now the unknown function. From the relation (1800) we derive

$$\frac{\partial\Omega}{\partial x}+\frac{\partial\Omega}{\partial F}\frac{\partial F}{\partial x}=0\,;\quad\ldots\ldots;\qquad\frac{\partial\Omega}{\partial t}+\frac{\partial\Omega}{\partial F}\frac{\partial F}{\partial t}=0\,;\qquad(1801)$$

and replacing $\partial F/\partial x, \ldots, \partial F/\partial t$ by the values from the preceding relations, equation (1799) becomes

$$\left.\begin{aligned}\left(\frac{1}{2}\frac{d\kappa}{dt}x+\beta y+\delta_1\right)\frac{\partial\Omega}{\partial x}+\left(\frac{1}{2}\frac{d\kappa}{dt}y-\beta x+\delta_2\right)\frac{\partial\Omega}{\partial y}&\\ +\left(\frac{1}{2}\frac{d\kappa}{dt}z+\delta_3\right)\frac{\partial\Omega}{\partial z}+\kappa\frac{\partial\Omega}{\partial t}-\Big\{\tfrac{1}{4}(x^2+y^2+z^2)\kappa\frac{d^3\kappa}{dt^3}&\\ +(x\ddot{\delta}_1+y\ddot{\delta}_2+z\ddot{\delta}_3)\kappa\Big\}\frac{\partial\Omega}{\partial F}=0\,.&\end{aligned}\right\}\tag{1802}$$

The new equation (1802) is seen to be a *homogeneous* linear partial differential equation for Ω in the five variables F, x, y, z, and t.

To solve equation (1802), we first write down the appropriate subsidiary equations, which are

$$\left.\begin{aligned}\frac{dx}{\frac{1}{2}\frac{d\kappa}{dt}x+\beta y+\delta_1}=\frac{dy}{\frac{1}{2}\frac{d\kappa}{dt}y-\beta x+\delta_2}=\frac{dz}{\frac{1}{2}\frac{d\kappa}{dt}z+\delta_3}=\frac{dt}{\kappa}&\\ =-\frac{dF}{\frac{1}{4}(x^2+y^2+z^2)\kappa\frac{d^3\kappa}{dt^3}+(x\ddot{\delta}_1+y\ddot{\delta}_2+z\ddot{\delta}_3)\kappa}\,.&\end{aligned}\right\}\tag{1803}$$

These equations can be expressed alternatively as

$$\left.\begin{aligned}\kappa\frac{dx}{dt}&=\frac{1}{2}\frac{d\kappa}{dt}x+\beta y+\delta_1\,,\\ \kappa\frac{dy}{dt}&=\frac{1}{2}\frac{d\kappa}{dt}y-\beta x+\delta_2\,,\\ \kappa\frac{dz}{dt}&=\frac{1}{2}\frac{d\kappa}{dt}z+\delta_3\,,\end{aligned}\right\}\tag{1804}$$

and

$$\frac{dF}{dt}+\tfrac{1}{4}(x^2+y^2+z^2)\frac{d^3\kappa}{dt^3}+(x\ddot{\delta}_1+y\ddot{\delta}_2+z\ddot{\delta}_3)=0\,.\tag{1805}$$

The equations (1804) are identical with the system (1668), which we have already encountered in § 73 while solving for the complemen-

tary function of equation (1665). As in § 73 we shall introduce the variables ξ, η, ζ, and ι defined by the relations

$$\kappa = \phi^2 \, ; \qquad \iota = \int^t \frac{dt}{\phi^2} \, , \tag{1806}$$

and

$$x = \xi\phi \, ; \qquad y = \eta\phi \, ; \qquad z = \zeta\phi \, . \tag{1807}$$

In terms of these variables the equations (1804) reduce to (cf. Eqs. [1673] and [1674])

$$\frac{d\xi}{d\iota} = \beta\eta + \frac{\delta_1}{\phi} \, , \qquad \frac{d\eta}{d\iota} = -\beta\xi + \frac{\delta_2}{\phi} \, ; \qquad \frac{d\zeta}{d\iota} = \frac{\delta_3}{\phi} \, . \tag{1808}$$

Further, equation (1805) is readily seen to be equivalent to

$$\frac{dF}{dt} + \tfrac{1}{4}(\xi^2 + \eta^2 + \zeta^2)\phi^2 \frac{d^3}{dt^3}\phi^2 + \phi(\xi\ddot{\delta}_1 + \eta\ddot{\delta}_2 + \zeta\ddot{\delta}_3) = 0 \, . \tag{1809}$$

Since

$$\phi^2 \frac{d^3}{dt^3}\phi^2 = \frac{d}{dt}\left(\phi^2 \frac{d^2}{dt^2}\phi^2 - \frac{1}{2}\left[\frac{d}{dt}\phi^2\right]^2\right) = 2\frac{d}{dt}(\phi^3\ddot{\phi}) \, , \tag{1810}$$

we can re-write equation (1809) in the form

$$\frac{dF}{d\iota} + \tfrac{1}{2}(\xi^2 + \eta^2 + \zeta^2)\frac{d}{d\iota}(\phi^3\ddot{\phi}) + \phi^3(\xi\ddot{\delta}_1 + \eta\ddot{\delta}_2 + \zeta\ddot{\delta}_3) = 0 \, . \tag{1811}$$

We have already solved the equations (1808) in § 73 and found that the solutions of these equations can be expressed in the form of three first-integrals:

$$\left.\begin{aligned} (\xi - \xi_0)^2 + (\eta - \eta_0)^2 &= \text{constant} \, , \\ \tan^{-1}\frac{\eta - \eta_0}{\xi - \xi_0} + \beta\iota &= \text{constant} \, , \\ \zeta - \zeta_0 &= \text{constant} \, , \end{aligned}\right\} \tag{1812}$$

where ξ_0, η_0, and ζ_0 are certain functions of ι determined in terms of the δ's by means of the relations (1713)–(1715).

To solve equation (1811), we proceed as follows: We first notice that

$$\frac{d}{dt} = \frac{1}{\phi^2}\frac{d}{d\iota}; \qquad \frac{d^2}{dt^2} = \frac{1}{\phi^4}\frac{d^2}{d\iota^2} - \frac{2}{\phi^3}\dot{\phi}\frac{d}{d\iota}. \tag{1813}$$

On the other hand, since we can write

$$\ddot{\delta}_1 = \frac{d^2}{dt^2}\left(\phi\frac{\delta_1}{\phi}\right) = \phi\frac{d^2}{dt^2}\left(\frac{\delta_1}{\phi}\right) + 2\dot{\phi}\frac{d}{dt}\left(\frac{\delta_1}{\phi}\right) + \ddot{\phi}\frac{\delta_1}{\phi}, \tag{1814}$$

we have (according to Eqs. [1813])

$$\ddot{\delta}_1 = \frac{1}{\phi^3}\frac{d^2}{d\iota^2}\left(\frac{\delta_1}{\phi}\right) + \ddot{\phi}\frac{\delta_1}{\phi}. \tag{1815}$$

We have similar relations for $\ddot{\delta}_2$ and $\ddot{\delta}_3$. Introducing these expressions for $\ddot{\delta}_1$, $\ddot{\delta}_2$, and $\ddot{\delta}_3$ into equation (1811), we have

$$\left.\begin{aligned} \frac{dF}{d\iota} + \tfrac{1}{2}(\xi^2+\eta^2+\zeta^2)\frac{d}{d\iota}(\phi^3\ddot{\phi}) + \phi^3\ddot{\phi}\left(\xi\frac{\delta_1}{\phi} + \eta\frac{\delta_2}{\phi} + \zeta\frac{\delta_3}{\phi}\right) \\ + \xi\frac{d^2}{d\iota^2}\left(\frac{\delta_1}{\phi}\right) + \eta\frac{d^2}{d\iota^2}\left(\frac{\delta_2}{\phi}\right) + \zeta\frac{d^2}{d\iota^2}\left(\frac{\delta_3}{\phi}\right) = 0. \end{aligned}\right\} \tag{1816}$$

From the equations (1808) we readily find that

$$\xi\frac{\delta_1}{\phi} + \eta\frac{\delta_2}{\phi} + \zeta\frac{\delta_3}{\phi} = \frac{1}{2}\frac{d}{d\iota}(\xi^2+\eta^2+\zeta^2). \tag{1817}$$

Combining equations (1816) and (1817), we have

$$\left.\begin{aligned} \frac{dF}{d\iota} + \frac{1}{2}\frac{d}{d\iota}([\xi^2+\eta^2+\zeta^2]\phi^3\ddot{\phi}) + \xi\frac{d^2}{d\iota^2}\left(\frac{\delta_1}{\phi}\right) + \eta\frac{d^2}{d\iota^2}\left(\frac{\delta_2}{\phi}\right) \\ + \zeta\frac{d^2}{d\iota^2}\left(\frac{\delta_3}{\phi}\right) = 0. \end{aligned}\right\} \tag{1818}$$

Again, using equations (1808), we find

$$\left.\begin{aligned}
&\xi \frac{d^2}{d\iota^2}\left(\frac{\delta_1}{\phi}\right) + \eta \frac{d^2}{d\iota^2}\left(\frac{\delta_2}{\phi}\right) \\
&\quad = \xi \frac{d^2}{d\iota^2}\left(\frac{d\xi}{d\iota} - \beta\eta\right) + \eta \frac{d^2}{d\iota^2}\left(\frac{d\eta}{d\iota} + \beta\xi\right) \\
&\quad = \xi \frac{d^3\xi}{d\iota^3} + \eta \frac{d^3\eta}{d\iota^3} - \beta\left(\xi \frac{d^2\eta}{d\iota^2} - \eta \frac{d^2\xi}{d\iota^2}\right) \\
&\quad = \frac{d}{d\iota}\left\{\xi \frac{d^2\xi}{d\iota^2} - \frac{1}{2}\left(\frac{d\xi}{d\iota}\right)^2 + \eta \frac{d^2\eta}{d\iota^2} - \frac{1}{2}\left(\frac{d\eta}{d\iota}\right)^2 - \beta\xi \frac{d\eta}{d\iota} + \beta\eta \frac{d\xi}{d\iota}\right\} \\
&\quad = \frac{d}{d\iota}\left\{\xi \frac{d}{d\iota}\left(\frac{d\xi}{d\iota} - \beta\eta\right) + \eta \frac{d}{d\iota}\left(\frac{d\eta}{d\iota} + \beta\xi\right) - \frac{1}{2}\left(\frac{d\xi}{d\iota}\right)^2 - \frac{1}{2}\left(\frac{d\eta}{d\iota}\right)^2\right\} \\
&\quad = \frac{d}{d\iota}\left\{\xi \frac{d}{d\iota}\left(\frac{\delta_1}{\phi}\right) + \eta \frac{d}{d\iota}\left(\frac{\delta_2}{\phi}\right) - \frac{1}{2}\left(\beta\eta + \frac{\delta_1}{\phi}\right)^2 - \frac{1}{2}\left(-\beta\xi + \frac{\delta_2}{\phi}\right)^2\right\} .
\end{aligned}\right\} \quad (1819)$$

Similarly we find that

$$\left.\begin{aligned}
\zeta \frac{d^2}{d\iota^2}\left(\frac{\delta_3}{\phi}\right) = \zeta \frac{d^3\zeta}{d\iota^3} &= \frac{d}{d\iota}\left\{\zeta \frac{d^2\zeta}{d\iota^2} - \frac{1}{2}\left(\frac{d\zeta}{d\iota}\right)^2\right\} \\
&= \frac{d}{d\iota}\left\{\zeta \frac{d}{d\iota}\left(\frac{\delta_3}{\phi}\right) - \frac{1}{2}\left(\frac{\delta_3}{\phi}\right)^2\right\} .
\end{aligned}\right\} \quad (1820)$$

Combining equations (1818), (1819), and (1820), we finally have

$$\left.\begin{aligned}
&\frac{dF}{d\iota} + \frac{1}{2}\frac{d}{d\iota}\{(\xi^2 + \eta^2 + \zeta^2)\phi^3\ddot{\phi}\} \\
&\quad + \frac{d}{d\iota}\left\{\xi \frac{d}{d\iota}\left(\frac{\delta_1}{\phi}\right) + \eta \frac{d}{d\iota}\left(\frac{\delta_2}{\phi}\right) + \zeta \frac{d}{d\iota}\left(\frac{\delta_3}{\phi}\right) - \frac{1}{2}\left(\beta\eta + \frac{\delta_1}{\phi}\right)^2\right. \\
&\qquad\qquad \left. - \frac{1}{2}\left(-\beta\xi + \frac{\delta_2}{\phi}\right)^2 - \frac{1}{2}\left(\frac{\delta_3}{\phi}\right)^2\right\} = 0 .
\end{aligned}\right\} \quad (1821)$$

Equation (1821) can be integrated as it stands, and we have

$$\left.\begin{aligned}
&F + \tfrac{1}{2}(\xi^2 + \eta^2 + \zeta^2)\phi^3\ddot{\phi} + \xi \frac{d}{d\iota}\left(\frac{\delta_1}{\phi}\right) + \eta \frac{d}{d\iota}\left(\frac{\delta_2}{\phi}\right) + \zeta \frac{d}{d\iota}\left(\frac{\delta_3}{\phi}\right) \\
&\quad - \frac{1}{2}\left(\beta\eta + \frac{\delta_1}{\phi}\right)^2 - \frac{1}{2}\left(-\beta\xi + \frac{\delta_2}{\phi}\right)^2 - \frac{1}{2}\left(\frac{\delta_3}{\phi}\right)^2 = \text{constant} .
\end{aligned}\right\} \quad (1822)$$

Equation (1822) can be expressed more conveniently in terms of the vectors

$$\boldsymbol{\rho} = (\xi, \eta, \zeta) \; ; \qquad \boldsymbol{\delta} = (\delta_1, \delta_2, \delta_3) \; ; \qquad \boldsymbol{\beta} = (0, 0, \beta) \, . \quad (1823)^{98}$$

We readily verify that equation (1822) is equivalent to

$$F + \tfrac{1}{2}\phi^3\ddot{\phi}\,|\boldsymbol{\rho}|^2 + \boldsymbol{\rho} \cdot \frac{d}{dt}\left(\frac{\boldsymbol{\delta}}{\phi}\right) - \frac{1}{2}\left|\boldsymbol{\rho} \times \boldsymbol{\beta} + \frac{\boldsymbol{\delta}}{\phi}\right|^2 = \text{constant} \, . \quad (1824)$$

Equations (1812) and (1824) represent the general solution of the subsidiary equations (1803) and define the necessary first-integrals. Consequently, the general solution of the partial differential equation (1802) can be written as

$$\left.\begin{aligned} \Omega(F, \xi, \eta, \zeta, t) \equiv \Omega \Big\{ & F + \tfrac{1}{2}\,\phi^3\ddot{\phi}\,|\boldsymbol{\rho}|^2 + \boldsymbol{\rho} \cdot \frac{d}{dt}\left(\frac{\boldsymbol{\delta}}{\phi}\right) - \frac{1}{2}\left|\boldsymbol{\rho} \times \boldsymbol{\beta} + \frac{\boldsymbol{\delta}}{\phi}\right|^2 ; \\ & (\xi - \xi_0)^2 + (\eta - \eta_0)^2 ; \; \tan^{-1}\frac{\eta - \eta_0}{\xi - \xi_0} + \beta t ; \; \zeta - \zeta_0 \Big\} \, , \\ & = 0 \end{aligned}\right\} \quad (1825)$$

where the quantity on the right-hand side stands for any arbitrary function of the arguments specified. It is now clear that we can express the solution (1825) alternatively in the form

$$\left.\begin{aligned} F + \tfrac{1}{2}\phi^3\ddot{\phi}\,|\boldsymbol{\rho}|^2 + \boldsymbol{\rho} \cdot \frac{d}{dt}\left(\frac{\boldsymbol{\delta}}{\phi}\right) - \frac{1}{2}\left|\boldsymbol{\rho} \times \boldsymbol{\beta} + \frac{\boldsymbol{\delta}}{\phi}\right|^2 & \\ = \mathfrak{P}^*\{(\xi - \xi_0)^2 + (\eta - \eta_0)^2 ; \; \vartheta + \beta t ; \; \zeta - \zeta_0\} \, , & \end{aligned}\right\} \quad (1826)$$

where $\mathfrak{P}^*$ denotes an arbitrary function of the arguments specified.[99] Hence,

$$\left.\begin{aligned} \mathfrak{P}(\xi, \eta, \zeta, t) &= \frac{1}{\phi^2} F(\xi, \eta, \zeta, t) \\ &= -\tfrac{1}{2}\phi\ddot{\phi}\,|\boldsymbol{\rho}|^2 - \frac{\boldsymbol{\rho}}{\phi^2} \cdot \frac{d}{dt}\left(\frac{\boldsymbol{\delta}}{\phi}\right) + \frac{1}{2\phi^2}\left|\boldsymbol{\rho} \times \boldsymbol{\beta} + \frac{\boldsymbol{\delta}}{\phi}\right|^2 \\ &\quad + \frac{1}{\phi^2}\,\mathfrak{P}^*\{(\xi - \xi_0)^2 + (\eta - \eta_0)^2 ; \; \vartheta + \beta t ; \; \zeta - \zeta_0\} \, , \end{aligned}\right\} \quad (1827)$$

[98] It will be recalled that, according to the choice of the orientation of our system of co-ordinates, the axis of rotation defined by the vector $\boldsymbol{\beta}$ is along the z-direction (cf. p. 580).

[99] As in § 73, we have used ϑ to denote the angle $\tan^{-1}(\eta - \eta_0)/(\xi - \xi_0)$ in equation (1826).

or, somewhat differently as (cf. Eqs. [1806] and [1807])

$$\left.\begin{aligned}\mathfrak{V} = -\frac{1}{2}\frac{\ddot{\phi}}{\phi}r^2 - \frac{1}{\phi}\frac{d}{dt}\left(\frac{\boldsymbol{\delta}}{\phi}\right)\cdot \boldsymbol{r} + \frac{1}{2\phi^4}\left|\boldsymbol{r}\times\boldsymbol{\beta}+\boldsymbol{\delta}\right|^2 \\ + \frac{1}{\phi^2}\mathfrak{V}^*\{(\xi-\xi_0)^2+(\eta-\eta_0)^2;\ \vartheta+\beta\iota;\ \zeta-\zeta_0\}\,.\end{aligned}\right\}\quad(1828)$$[100]

In the foregoing equation the term in $\mathfrak{V}^*$ represents the complementary function of equation (1665). Thus the most general particular integral of this equation is given by

$$\mathfrak{V}_p\ (\text{general}) = -\frac{1}{2}\frac{\ddot{\phi}}{\phi}r^2 - \frac{1}{\phi}\frac{d}{dt}\left(\frac{\boldsymbol{\delta}}{\phi}\right)\cdot\boldsymbol{r} + \frac{1}{2\phi^4}\left|\boldsymbol{r}\times\boldsymbol{\beta}+\boldsymbol{\delta}\right|^2.\quad(1829)$$

The general solution for the density function σ is now readily obtained. According to equation (1609), we have

$$\tfrac{1}{2}\sigma = \phi^2\mathfrak{V} + \tfrac{1}{2}\phi\ddot{\phi}r^2 + \phi\frac{d}{dt}\left(\frac{\boldsymbol{\delta}}{\phi}\right)\cdot\boldsymbol{r} - \frac{1}{2\phi^2}\left|\boldsymbol{r}\times\boldsymbol{\beta}+\boldsymbol{\delta}\right|^2.\quad(1830)$$

Using the solution (1828) for $\mathfrak{V}$, we have

$$\tfrac{1}{2}\sigma = \mathfrak{V}^*\{(\xi-\xi_0)^2+(\eta-\eta_0)^2;\ \vartheta+\beta\iota;\ \zeta-\zeta_0\}\,.\quad(1831)$$

In other words,

$$\sigma \equiv \sigma\{(\xi-\xi_0)^2+(\eta-\eta_0)^2;\ \vartheta+\beta\iota;\ \zeta-\zeta_0\}\,.\quad(1832)$$

[100] A consequence of this solution for stellar systems in *steady states* may be noted. Under steady-state circumstances δ_1, δ_2, δ_3, and ϕ are constants, and there is clearly no loss of generality if we set $\delta_1 = \delta_2 = 0$. (This corresponds to an appropriate choice of the *origin* of our system of co-ordinates [see Fig. 1 in Part IV].) Equation (1828) now reduces to

$$\mathfrak{V} = \mathfrak{V}\left(x^2+y^2;\ \theta+\frac{\beta}{\phi^2}t;\ z-\frac{\delta_3}{\phi^2}t\right),\quad(\text{i})$$

or, eliminating t between the last two integrals in (i), we have

$$\mathfrak{V} \equiv \mathfrak{V}\left(x^2+y^2;\ z+\frac{\delta_3}{\beta}\theta\right).\quad(\text{ii})$$

We have thus recovered the *helical symmetry* of $\mathfrak{V}$ for stellar systems in steady states proved in Part IV.

The foregoing equation represents the complete generalization of our earlier results in § 74 (subsection III, Eq. [1761]). While in § 74 the choice of a particular integral of a certain special form resulted in the explicit specification of the (ξ_0, η_0) locus and ζ_0 in terms of ι, these are now expressed in the form of certain indefinite integrals involving the functions δ_1/ϕ, δ_2/ϕ, and δ_3/ϕ, which are left entirely arbitrary. However, the general characteristics of the trajectories described by points of constant relative density σ can be traced by the methods we have already developed (pp. 596 and 617 and Fig. 10).

We thus see that the general method of interpreting the forms of extragalactic objects outlined in § 75 can be still further generalized. While some of the more detailed identifications suggested in § 75 are largely tentative, it appears that our expectation that the solution to the problem spiral structure in nebulae is not beyond the range of our fundamental kinematical postulates is amply justified.

XIV. THE PRINCIPLE OF SUPERPOSITION OF STELLAR SYSTEMS

77. *The statement of the problem.*—In current formulations of the problems of stellar dynamics the emphasis has always been on finding solutions of the equation of continuity regarded as a partial differential equation for the distribution function $\Psi(x, y, z, U, V, W, t)$. Now, the equation of continuity

$$\frac{\partial\Psi}{\partial t} + U\frac{\partial\Psi}{\partial x} + V\frac{\partial\Psi}{\partial y} + W\frac{\partial\Psi}{\partial z} - \frac{\partial\mathfrak{B}}{\partial x}\frac{\partial\Psi}{\partial U} - \frac{\partial\mathfrak{B}}{\partial y}\frac{\partial\Psi}{\partial V} - \frac{\partial\mathfrak{B}}{\partial z}\frac{\partial\Psi}{\partial W} = 0 \qquad (1833)$$

is linear in Ψ; and consequently, if Ψ_1 and Ψ_2 are two distinct solutions of equation (1833), then $\Psi_1 + \Psi_2$ is also a solution. To be more specific, let

$$\Psi_i = \Psi_i(Q_i + \sigma_i) \qquad (i = 1, \dots, n)\,, \qquad (1834)$$

be n distinct solutions of the equation of continuity. In equation (1834) the Q_i's are general quadratic forms in the variables $U - U_i$, $V - V_i$, and $W - W_i$; further, the coefficients of the various quad-

ratic forms, the motions (U_i, V_i, W_i), and the σ_i's are all functions of x, y, z, and t. Then

$$\Psi = \sum_{i=1}^{n} \Psi_i \tag{1835}$$

is also a solution of the equation of continuity, provided certain *conditions of consistency* are satisfied. The reason for the existence of such conditions is easily understood: According to equation (1835), the stellar system can be regarded as consisting of n subsystems, described, respectively, by Ψ_1, Ψ_2, ... , and Ψ_n. Since, however, the motions in each of the subsystems are governed by the same gravitational potential $\mathfrak{B}(x, y, z, t)$, it is clear that there should be certain restrictions in order that there be no inconsistency resulting from the superposition of the different subsystems. The problem which thus presents itself can be formulated as follows:

What are the circumstances under which we can regard a stellar system as consisting of two or more independent subsystems each of which satisfies our fundamental kinematical postulates?

In §§ 78 and 79 certain special examples of stellar systems formed by superposition are considered, and the precise forms which the conditions of consistency take for these cases are obtained. In §§ 80 and 81 the problem of superposition is considered from a more general point of view.

78. *Superposition of two-dimensional stellar systems in steady states.*—In Part II we have shown that, for two-dimensional stellar systems under steady-state conditions, $\mathfrak{B}$ must necessarily be characterized by circular symmetry.[101] Also, if the compatibility conditions are to restrict $\mathfrak{B}$ no further, then the solution

$$\Psi = \Psi(Q + \sigma) \, , \tag{1836}$$

takes the form (cf. Eqs. [94] and [108])

$$\left.\begin{aligned} Q + \sigma = a(U - U_0)^2 + 2h(U - U_0)(V - V_0) \\ + b(V - V_0)^2 + \sigma \, , \end{aligned}\right\} \tag{1837}$$

[101] I.e., apart from the physically unimportant cases $\beta = 0$ and $\beta = \delta_1 = \delta_2 = 0$ (cf. the theorem stated at the end of § 8).

where

$$\left.\begin{aligned} a &= \kappa_1 + \kappa_2 y^2 \,; \qquad b = \kappa_1 + \kappa_2 x^2 \,; \qquad h = -\kappa_2 xy \\ U_0 &= \frac{\beta y}{\kappa_1 + \kappa_2(x^2 + y^2)} \,; \qquad V_0 = -\frac{\beta x}{\kappa_1 + \kappa_2(x^2 + y^2)} \,, \end{aligned}\right\} \tag{1838}$$

where κ_1, κ_2, and β are constants. The compatibility conditions (76) and (77) now reduce to

$$\left.\begin{aligned} (ax + hy)\frac{d\mathfrak{V}}{d\tau} &= -\frac{1}{2}\frac{\partial\chi}{\partial x} \,, \\ (hx + by)\frac{d\mathfrak{V}}{d\tau} &= -\frac{1}{2}\frac{\partial\chi}{\partial y} \,, \end{aligned}\right\} \tag{1839}$$

where τ has the same meaning as in Part II (Eq. [80]). Using equation (107), the foregoing equations become

$$\kappa_1 x\frac{d\mathfrak{V}}{d\tau} = -\frac{1}{2}\frac{\partial\chi}{\partial x} \,; \qquad \kappa_1 y\frac{d\mathfrak{V}}{d\tau} = -\frac{1}{2}\frac{\partial\chi}{\partial y} \,. \tag{1840}$$

Hence, χ is also circularly symmetrical, and equations (1840) are equivalent to

$$\kappa_1\frac{d\mathfrak{V}}{d\tau} = -\frac{1}{2}\frac{d\chi}{d\tau} \,, \tag{1841}$$

or, after integration,

$$-\tfrac{1}{2}\chi = \kappa_1\mathfrak{V} + \text{constant} \,. \tag{1842}$$

Since (cf. Eqs. [54] and [55])

$$-\chi = aU_0^2 + 2hU_0V_0 + bV_0^2 + \sigma \,, \tag{1843}$$

we have, according to equation (1838),

$$-\chi = \frac{\beta^2\bar{\omega}^2}{\kappa_1 + \kappa_2\bar{\omega}^2} + \sigma + \text{constant} \,. \tag{1844}$$

Combining equations (1842) and (1844), we have

$$2\mathfrak{V} = \frac{\beta^2\bar{\omega}^2}{\kappa_1(\kappa_1 + \kappa_2\bar{\omega}^2)} + \frac{\sigma}{\kappa_1} + \text{constant} \,. \tag{1845}$$

Let us now consider a stellar system formed by the superposition of two stellar systems each of which is similar to one we have just considered. We accordingly assume that

$$\Psi = \Psi_1(Q_1 + \sigma_1) + \Psi_2(Q_2 + \sigma_2) \,, \tag{1846}$$

where (cf. Eqs. [1837] and [1838])

$$\left.\begin{aligned} Q_1 + \sigma_1 &= (\kappa_{11} + \kappa_{12}y^2)(U - U_1)^2 - 2\kappa_{12}xy(U - U_1)(V - V_1) \\ &\quad + (\kappa_{11} + \kappa_{12}x^2)(V - V_1)^2 + \sigma_1 \,, \\ Q_2 + \sigma_2 &= (\kappa_{21} + \kappa_{22}y^2)(U - U_2)^2 - 2\kappa_{22}xy(U - U_2)(V - V_2) \\ &\quad + (\kappa_{21} + \kappa_{22}x^2)(V - V_2)^2 + \sigma_2 \,, \end{aligned}\right\} \tag{1847}$$

$$\left.\begin{aligned} U_1 &= \frac{\beta_1 y}{\kappa_{11} + \kappa_{12}\bar{\omega}^2} \,; \qquad V_1 = -\frac{\beta_1 x}{\kappa_{11} + \kappa_{12}\bar{\omega}^2} \,, \\ U_2 &= \frac{\beta_2 y}{\kappa_{21} + \kappa_{22}\bar{\omega}^2} \,; \qquad V_2 = -\frac{\beta_2 x}{\kappa_{21} + \kappa_{22}\bar{\omega}^2} \,, \end{aligned}\right\} \tag{1848}$$

where κ_{11}, κ_{12}, κ_{21}, κ_{22}, β_1, and β_2 are all constants. Defined in this manner, Ψ_1 and Ψ_2 are both solutions of the two-dimensional equation of continuity (for a steady state) provided (cf. Eq. [1845])

$$2\mathfrak{V} = \frac{\beta_1^2\bar{\omega}^2}{\kappa_{11}(\kappa_{11} + \kappa_{12}\bar{\omega}^2)} + \frac{\sigma_1}{\kappa_{11}} + c_1 \tag{1849}$$

and

$$2\mathfrak{V} = \frac{\beta_2^2\bar{\omega}^2}{\kappa_{21}(\kappa_{21} + \kappa_{22}\bar{\omega}^2)} + \frac{\sigma_2}{\kappa_{21}} + c_2 \,, \tag{1850}$$

where c_1 and c_2 are constants. Hence,

$$c_1 + \frac{\sigma_1}{\kappa_{11}} + \frac{\beta_1^2\bar{\omega}^2}{\kappa_{11}(\kappa_{11} + \kappa_{12}\bar{\omega}^2)} = c_2 + \frac{\sigma_2}{\kappa_{21}} + \frac{\beta_2^2\bar{\omega}^2}{\kappa_{21}(\kappa_{21} + \kappa_{22}\bar{\omega}^2)} \,, \tag{1851}$$

which is essentially a relation between the density distributions in the two subsystems. Equation (1851) is therefore the condition which should be satisfied in order that the motions in the two subsystems may be governed by the same gravitational potential $\mathfrak{V}$.

The physical meaning of the relation (1851) becomes somewhat clearer if we consider the special case

$$\Psi_1 = e^{-(Q_1+\sigma_1)} \,; \qquad \Psi_2 = e^{-(Q_2+\sigma_2)} \,. \tag{1852}$$

Under these circumstances

$$\mathfrak{N}_1 = \frac{\pi}{\sqrt{\kappa_{11}(\kappa_{11} + \kappa_{12}\bar{\omega}^2)}} e^{-\sigma_1}; \qquad \mathfrak{N}_2 = \frac{\pi}{\sqrt{\kappa_{21}(\kappa_{21} + \kappa_{22}\bar{\omega}^2)}} e^{-\sigma_2}, \quad (1853)$$

where $\mathfrak{N}_1$ and $\mathfrak{N}_2$ are the number of stars per unit area in the two subsystems. Eliminating σ_1 and σ_2 between the relations (1851) and (1853), we readily obtain

$$\left.\begin{aligned} &\left[\mathfrak{N}_1\sqrt{(\kappa_{11} + \kappa_{12}\bar{\omega}^2)}\; e^{-(\beta_1^2\bar{\omega}^2)/(\kappa_{11}+\kappa_{12}\bar{\omega}^2)}\right]^{1/\kappa_{11}} \\ &\qquad = \text{constant}\left[\mathfrak{N}_2\sqrt{(\kappa_{21} + \kappa_{22}\bar{\omega}^2)}\; e^{-(\beta_2^2\bar{\omega}^2)/(\kappa_{21}+\kappa_{22}\bar{\omega}^2)}\right]^{1/\kappa_{21}}. \end{aligned}\right\} (1854)$$

So far we have considered the result of the superposition of only two subsystems. The extension to more than two systems is, of course, immediate. It is further clear that for a stellar system formed by the superposition of n distinct subsystems there will be $(n - 1)$ consistency relations of the form (1851).

We should finally remark that equation (1845) is equivalent to a relation which Oort has used on several occasions in the past.[102] It would consequently be of interest to consider applications of relations of the form (1851) or (1854).

79. *Superposition of axially symmetrical stellar systems in steady states.*—We have enumerated in § 59 the two special cases of axially symmetrical stellar systems for which the appropriate integrability conditions are "trivially" satisfied. We shall now consider the problem of superposition for these two cases.

Case (i).—For this case it is readily found that (cf. Eq. [1312])

$$2\kappa_1\mathfrak{V} = \sigma + \frac{\beta^2\bar{\omega}^2}{\kappa_1 + \kappa_2\bar{\omega}^2} + \text{constant}, \qquad (1855)$$

which is seen to be formally identical with equation (1845). Consequently, for the superposition of two such stellar systems the condition of consistency takes the same form as equation (1851). Equation (1854) is also valid as it stands if we now interpret $\mathfrak{N}_1$ and $\mathfrak{N}_2$ as the number of stars per unit volume in the two subsystems.

Case (ii).—For this case we have (cf. Eq. [1329])

$$\sigma = -\frac{\beta^2\bar{\omega}^2}{\kappa_1 + \kappa_2\bar{\omega}^2} + 2\kappa_1\mathfrak{V}_1(\bar{\omega}) + 2\kappa_3\mathfrak{V}_2(z) + \text{constant}. \quad (1856)$$

[102] E.g., see *Ap. J.*, **91**, 273, 1940.

We can therefore write

$$\sigma = \sigma_1(\bar{\omega}) + \sigma_2(z) , \tag{1857}$$

where

$$\left.\begin{aligned} \sigma_1 &= -\frac{\beta^2\bar{\omega}^2}{\kappa_1 + \kappa_2\bar{\omega}^2} + 2\kappa_1\mathfrak{V}_1(\bar{\omega}) + c_1 , \\ \sigma_2 &= 2\kappa_3\mathfrak{V}_2(z) + c_2 , \end{aligned}\right\} \tag{1858}$$

where c_1 and c_2 are constants. For the superposition of two such systems the necessary conditions are (in an obvious notation)

$$\left.\begin{aligned} \frac{\sigma_{11}}{\kappa_{11}} + \frac{\beta_1^2\bar{\omega}^2}{\kappa_{11}(\kappa_{11} + \kappa_{12}\bar{\omega}^2)} + c_{11} &= \frac{\sigma_{21}}{\kappa_{21}} + \frac{\beta_2^2\bar{\omega}^2}{\kappa_{21}(\kappa_{21} + \kappa_{22}\bar{\omega}^2)} + c_{21} , \\ \frac{\sigma_{12}}{\kappa_{13}} + c_{12} &= \frac{\sigma_{22}}{\kappa_{23}} + c_{22} . \end{aligned}\right\} \tag{1859}$$

The foregoing conditions have been obtained for the superposition of two systems. But the conditions are readily extended for the superposition of more than two systems.

80. *The principle of superposition for stellar systems in steady states.*—We consider a stellar system formed by the superposition of several subsystems each of which is in a steady state. It is clear that the differential equations for the coefficients of the velocity ellipsoid and for the Δ's for the different subsystems are the same. Hence, the solutions for these quantities for the different subsystems have similar forms but are expressed in terms of different sets of constants of integration.

The conditions on the solutions for $\mathbf{\Delta}$ are readily found. According to the fundamental theorem of stellar dynamics proved in Part IV, $\mathfrak{V}$ has a helical symmetry about an axis parallel to the $\boldsymbol{\beta}$ direction and passing through the point $(\boldsymbol{\delta} \times \boldsymbol{\beta})/|\boldsymbol{\beta}|^2$ (see Fig. 1). Further, the constant of the helical symmetry is $(\boldsymbol{\beta} \cdot \boldsymbol{\delta})/|\boldsymbol{\beta}|^2$. Since the same gravitational potential $\mathfrak{V}$ governs the motions in all the subsystems, it follows that the quantities $(\boldsymbol{\beta} \times \boldsymbol{\delta})/|\boldsymbol{\beta}|^2$ and $(\boldsymbol{\beta} \cdot \boldsymbol{\delta})/|\boldsymbol{\beta}|^2$ should be the same for all the different subsystems. These conditions are readily seen to imply that (i) the vector $\boldsymbol{\beta}$ defines a unique direction and is the same for all the subsystems; (ii) the plane defined by vectors $\boldsymbol{\beta}$ and $\boldsymbol{\delta}$ is an *invariable plane* of the system and is

therefore the same for the subsystems; (iii) the angle between the vectors $\boldsymbol{\beta}$ and $\boldsymbol{\delta}$ and the ratios $|\boldsymbol{\delta}|/|\boldsymbol{\beta}|$ are the same for all the subsystems.

The conditions on the coefficients of the velocity ellipsoid and σ cannot be stated so explicitly. Each subsystem provides a matrix equation of the form

$$\boldsymbol{A} \operatorname{grad} \mathfrak{V} = -\tfrac{1}{2} \operatorname{grad} \chi \,, \tag{1860}$$

where $\boldsymbol{A}$ is the matrix associated with the fundamental quadratic form (see Eq. [955]). Equation (1860) requires, of course, that

$$\operatorname{curl} (\boldsymbol{A} \operatorname{grad} \mathfrak{V}) = 0 \,. \tag{1861}$$

Let us suppose that the integrability conditions (1861) are satisfied. Now, according to (1860),

$$\operatorname{grad} \mathfrak{V} = -\tfrac{1}{2}\boldsymbol{A}^{-1} \operatorname{grad} \chi \,, \tag{1862}$$

where $\boldsymbol{A}^{-1}$ is the reciprocal matrix of $\boldsymbol{A}$. Each subsystem provides an equation for $\mathfrak{V}$ of the form (1862). Consequently,

$$\boldsymbol{A}^{-1} \operatorname{grad} \chi \tag{1863}$$

for the different subsystems should be the same. This is the principle of superposition for stellar systems in steady states.

81. *Superposition of stellar systems in nonsteady states.*—For stellar systems in nonsteady states the principle of superposition can be stated even less explicitly than for the case of steady states (§ 80). We have first to suppose that the integrability conditions (958) and (960) are satisfied for each of the subsystems. We should then arrange that the properties implied for $\mathfrak{V}$ according to the solutions for $\boldsymbol{A}$ and $\boldsymbol{\Delta}$ for the different subsystems are identical. Further, according to equation (956), we should also arrange that

$$\boldsymbol{A}^{-1} \left(\frac{\partial \boldsymbol{\Delta}}{\partial t} + \tfrac{1}{2} \operatorname{grad} \chi \right) \tag{1864}$$

for the different subsystems are all the same.

In any special case the conditions stated in the preceding paragraph can easily be fulfilled. Thus, as an illustrative example let us

consider the superposition of spiral systems of the kind considered in Part XIII.

Let us first consider the case $\boldsymbol{\delta} \equiv 0$ (§§ 71 and 72) and restrict ourselves further to the superposition of only two such systems. From equation (1631) it now follows that the ϕ's for the two subsystems are proportional to each other:

$$\frac{\phi_1}{\phi_2} = \gamma = \text{constant}\,. \tag{1865}$$

Further, we should also have

$$\frac{\beta_1}{\beta_2} = \frac{\phi_1^2}{\phi_2^2} = \gamma^2 = \text{constant}\,. \tag{1866}$$

Finally, the relations between the σ's for the two subsystems can be readily written down by eliminating $\mathfrak{B}$ between the two relations of the form (1634). We find

$$c_1 + \frac{\sigma_1}{\phi_1^2} = c_2 + \frac{\sigma_2}{\phi_2^2}\,, \tag{1867}$$

where c_1 and c_2 are constants. It may be noted that spiral systems superposed in this manner are all characterized by the same angle α (cf. Eqs. [1655] and [1866]).

For the general case considered in §§ 73 and 74 we have similar results. Again, considering the superposition of two such systems, it is seen that the ratios of the corresponding q's for the two subsystems are also in the ratio (1865); further, the basic δ- and the (ξ_0, η_0) loci are the same for the two subsystems.[103] It is thus seen that in the theory of spiral systems considered in Part XIII we can incorporate the phenomenon of star-streaming in the sense of Kapteyn, since the distribution function is now expressed as the sum of two spherical distributions with respect to local standards, which are different for the two systems.

YERKES OBSERVATORY
May 1940

[103] These results can, of course, be extended quite readily for the superposition of more than two systems.

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VOLUME 92 NUMBER 3

THE ASTROPHYSICAL JOURNAL

AN INTERNATIONAL REVIEW OF SPECTROSCOPY AND ASTRONOMICAL PHYSICS

Founded in 1895 by GEORGE E. HALE *and* JAMES E. KEELER

NOVEMBER 1940

THE UNIVERSITY OF CHICAGO PRESS
CHICAGO, ILLINOIS, U.S.A.

November 1940 Vol. 92, No. 3

THE ASTROPHYSICAL JOURNAL

AN INTERNATIONAL REVIEW OF SPECTROSCOPY AND ASTRONOMICAL PHYSICS

The Astrophysical Journal is published bimonthly by the University of Chicago at the University of Chicago Press, 5750 Ellis Avenue, Chicago, Illinois, during July, September, November, January, March, and May. ¶The subscription price is $10.00 a year; the price of single copies is $2.00. Orders for service of less than a full year will be charged at the single-copy rate. ¶Postage is prepaid by the publishers on all orders from the United States and its possessions, Argentina, Bolivia, Brazil, Chile, Colombia, Costa Rica, Cuba, Dominican Republic, Ecuador, Guatemala, Haiti, Republic of Honduras, Mexico, Morocco (Spanish Zone), Nicaragua, Panama, Paraguay, Peru, Rio de Oro, El Salvador, Spain (including Balaeric Islands, Canary Islands, and the Spanish Offices in Northern Africa; Andorra), Spanish Guinea, Uruguay, and Venezuela. ¶Postage is charged extra as follows: for Canada and Newfoundland, 42 cents on annual subscriptions (total $10.42); on single copies, 7 cents (total $2.07); for all other countries in the Postal Union, 96 cents on annual subscriptions (total $10.96), on single copies 16 cents (total $2.16). ¶Patrons are requested to make all remittances payable to The University of Chicago Press, in United States currency or its equivalent by postal or express money orders or bank drafts.

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Claims for missing numbers should be made within the month following the regular month of publication. The publishers expect to supply missing numbers free only when losses have been sustained in transit, and when the reserve stock will permit.

Business correspondence should be addressed to The University of Chicago Press, Chicago, Illinois.

Communications for the editors and manuscripts should be addressed to: Otto Struve, Editor of The Astrophysical Journal, Yerkes Observatory, Williams Bay, Wisconsin.

One copy of the corrected galley proof should be returned as soon as possible to the editor, Yerkes Observatory, Williams Bay, Wisconsin. Authors should take notice that the manuscript will not be sent to them with the proof.

The cable address is "Observatory, Williamsbay, Wisconsin."

The articles in this journal are indexed in the *International Index to Periodicals*, New York, N.Y.

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Entered as second-class matter, July 31, 1940, at the Post-Office, at Chicago, Ill., under the act of March 3, 1879.
Acceptance for mailing at special rate of postage provided for in United States Postal Act of October 3, 1917, Section 1103, amended February 28, 1925.

PRINTED IN THE U.S.A.